2.5

MATRIX FACTORIZATION

Linear Algebra
And Its Applications
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A factorization of a matrix $A$ is an equation that expresses $A$ as a product of two or more matrices. Whereas matrix multiplication involves a synthesis of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an analysis of data.
The LU factorization, described on the next few slides, is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

\[ Ax = b_1, \ Ax = b_2, \ldots, \ Ax = b_p \]  

(1)

When \( A \) is invertible, one could compute \( A^{-1} \) and then compute \( A^{-1}b_1, A^{-1}b_2, \) and so on.

However, it is more efficient to solve the first equation in the sequence (1) by row reduction and obtain the LU factorization of \( A \) at the same time. Thereafter, the remaining equations in sequence (1) are solved with the LU factorization.
THE LU FACTORIZATION

- At first, assume that $A$ is an $m \times n$ matrix that can be row reduced to echelon form, \textit{without row interchanges}.
- Then $A$ can be written in the form $A = LU$, were $L$ is an $m \times m$ lower triangular matrix with 1’s on the diagonal and $U$ is an $m \times n$ echelon form of $A$.
- For instance, see Fig. 1 below. Such a factorization is called an \textbf{LU factorization} of $A$. The matrix $L$ is invertible and is called a unit lower triangular matrix.

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & \end{bmatrix} \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 0 & 0 & \end{bmatrix} \]
Before studying how to construct $L$ and $U$, we should look at why they are so useful. When $A = LU$, the equation $Ax = b$ can be written as $L(Ux) = b$. Writing $y$ for $Ux$, we can find $x$ by solving the pair of equations

\[
\begin{align*}
Ly &= b \\
Ux &= y
\end{align*}
\]

First solve $Ly = b$ for $y$, and then solve $Ux = y$ for $x$. See Fig. 2 on the next slide. Each equation is easy to solve because $L$ and $U$ are triangular.
Example 1  It can be verified that

\[ A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 8 & 3 & 1 \\ 2 & -5 & 1 & 0 \\ -9 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU \]

Use this factorization of \( A \) to solve \( Ax = b \), where \( b = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix} \)
THE LU FACTORIZATION

Solution The solution of $Ly = b$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5.

\[
\begin{bmatrix} L & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & y \end{bmatrix}
\]

Then, for $Ux = y$, the “backward” phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.
The LU Factorization

- For instance, creating the zeros in column 4 of \([U \ y]\) requires 1 division in row 4 and 3 multiplication-addition pairs to add multiples of row 4 to the rows above.

\[
\begin{bmatrix}
U & y
\end{bmatrix} = \begin{bmatrix}
3 & -7 & -2 & 2 & -9 \\
0 & -2 & -1 & 2 & -4 \\
0 & 0 & -1 & 1 & 5 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 0 & -6 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}, \quad x = \begin{bmatrix}
3 \\
4 \\
-6 \\
-1
\end{bmatrix}
\]

- To find \(x\) requires 28 arithmetic operations, or “flops” (floating point operations), excluding the cost of finding \(L\) and \(U\). In contrast, row reduction of \([A \ b]\) to \([I \ x]\) takes 62 operations.
AN LU FACTORIZATION ALGORITHM

- Suppose $A$ can be reduced to an echelon form $U$ using only row replacements that add a multiple of one row to another below it.

- In this case, there exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that
  $$E_p \ldots E_1 A = U$$

- Then
  $$A = (E_p \ldots E_1)^{-1} U = LU$$  \hspace{1cm} (3)

- where
  $$L = (E_p \ldots E_1)^{-1}$$ \hspace{1cm} (4)

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus $L$ is unit lower triangular.
AN LU FACTORIZATION ALGORITHM

Note that row operations in equation (3), which reduce \( A \) to \( U \), also reduce the \( L \) in equation (4) to \( I \), because
\[
E_p \ldots E_1 L = (E_p \ldots E_1)(E_p \ldots E_1)^{-1} = I.
\]
This observation is the key to constructing \( L \).

Algorithm for an LU Factorization

1. Reduce \( A \) to an echelon form \( U \) by a sequence of row replacement operations, if possible.
2. Place entries in \( L \) such that the same sequence of row operations reduces \( L \) to \( I \).
AN LU FACTORIZATION ALGORITHM

- Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists.
- Example 2 on the followings slides will show how to implement step 2. By construction, $L$ will satisfy

$$(E_p \ldots E_1)L = I$$

- using the same $E_p, \ldots, E_1$ as in equation (3). Thus $L$ will be invertible, by the Invertible Matrix Theorem, with $(E_p \ldots E_1) = L^{-1}$. From (3), $L^{-1}A = U$, and $A = LU$. So step 2 will produce an acceptable $L$. 
Example 2  Find an LU factorization of

\[
A = \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
-4 & -5 & 3 & -8 & 1 \\
2 & -5 & -4 & 1 & 8 \\
-6 & 0 & 7 & -3 & 1 \\
\end{bmatrix}
\]

Solution  Since \( A \) has four rows, \( L \) should be \( 4 \times 4 \). The first column of \( L \) is the first column of \( A \) divided by the top pivot entry:

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-3 & -3 & 1 & 1 \\
\end{bmatrix}
\]
AN LU FACTORIZATION ALGORITHM

- Compare the first columns of $A$ and $L$. The row operations that create zeros in the first column of $A$ will also create zeros in the first column of $L$.

- To make this same correspondence of row operations on $A$ hold for the rest of $L$, watch a row reduction of $A$ to an echelon form $U$. That is, highlight the entries in each matrix that are used to determine the sequence of row operations that transform $A$ onto $U$.

\[
A = \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
-4 & -5 & 3 & -8 & 1 \\
2 & -5 & -4 & 1 & 8 \\
-6 & 0 & 7 & -3 & 1
\end{bmatrix} \sim \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & -9 & -3 & -4 & 10 \\
0 & 12 & 4 & 12 & -5
\end{bmatrix} = A_1
\]

\[
\sim A_2 = \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 4 & 7
\end{bmatrix} \sim \begin{bmatrix}
2 & 4 & -1 & 5 & -2 \\
0 & 3 & 1 & 2 & -3 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix} = U
\]
AN LU FACTORIZATION ALGORITHM

- The highlighted entries above determine the row reduction of $A$ to $U$. At each pivot column, divide the highlighted entries by the pivot and place the result onto $L$:

\[
\begin{bmatrix}
2 \\
-4 \\
2 \\
-6
\end{bmatrix}
\begin{bmatrix}
3 \\
-9 \\
12 \\
4
\end{bmatrix}
\begin{bmatrix}
2 \\
3 \\
2 \\
5
\end{bmatrix}
\]

\[\div 2 \quad \div 3 \quad \div 2 \quad \div 5\]

\[
\begin{bmatrix}
1 \\
-2 \\
1 \\
-3
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
-3 \\
4
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

- An easy calculation verifies that this $L$ and $U$ satisfy $LU = A$. 