2.3 CHARACTERIZATIONS OF INVERTIBLE MATRICES

\[
\begin{align*}
(A^T)^{-1} &= (A^{-1})^T \\
(A^{-1})^T A^T &= I \\
A A^{-1} &= I^T = I
\end{align*}
\]
The Invertible Matrix Theorem

Theorem 8: Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given $A$, the statements are either all true or all false.

a. $A$ is an invertible matrix.
b. $A$ is row equivalent to the $n \times n$ identity matrix.
c. $A$ has $n$ pivot positions.
d. The equation $Ax = 0$ has only the trivial solution.
e. The columns of $A$ form a linearly independent set.
f. The linear transformation $x \mapsto Ax$ is one-to-one.

g. The equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix $C$ such that $CA = I$.

k. There is an $n \times n$ matrix $D$ such that $AD = I$.

l. $A^T$ is an invertible matrix. $\begin{bmatrix} A & D \end{bmatrix} = I$
First, we need some notation.

If the truth of statement (a) always implies that statement (j) is true, we say that (a) implies (j) and write (a) ⇒ (j).

The proof will establish the “circle” of implications as shown in the following figure.

If any one of these five statements is true, then so are the others.
Finally, the proof will link the remaining statements of the theorem to the statements in this circle.

Proof: If statement (a) is true, then $A^{-1}$ works for $C$ in (j), so $(a) \implies (j)$.

Next, $(j) \implies (d)$.

Also, $(d) \implies (c)$.

If $A$ is square and has $n$ pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of $A$ is $I_n$.

Thus $(c) \implies (b)$.

Also, $(b) \implies (a)$.
THE INVERTIBLE MATRIX THEOREM

- This completes the circle in the previous figure.
- Next, \((a) \Rightarrow (k)\) because \(A^{-1}\) works for \(D\).
- Also, \((k) \Rightarrow (g)\) and \((g) \Rightarrow (a)\).
- So \((k)\) and \((g)\) are linked to the circle.
- Further, \((g)\), \((h)\), and \((i)\) are equivalent for any matrix.
- Thus, \((h)\) and \((i)\) are linked through \((g)\) to the circle.
- Since \((d)\) is linked to the circle, so are \((e)\) and \((f)\), because \((d)\), \((e)\), and \((f)\) are all equivalent for any matrix \(A\).
- Finally, \((a) \Rightarrow (l)\) and \((l) \Rightarrow (a)\).
- This completes the proof.
THE INVERTIBLE MATRIX THEOREM

- Theorem 8 could also be written as “The equation \( Ax = b \) has a unique solution for each \( b \) in \( \mathbb{R}^n \).”
- This statement implies (b) and hence implies that \( A \) is invertible.
- The following fact follows from Theorem 8.
  Let \( A \) and \( B \) be square matrices. If \( AB = I \), then \( A \) and \( B \) are both invertible, with \( B = A^{-1} \) and \( A = B^{-1} \).
- The Invertible Matrix Theorem divides the set of all \( n \times n \) matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.
THE INVERTIBLE MATRIX THEOREM

- Each statement in the theorem describes a property of every $n \times n$ invertible matrix.

- The negation of a statement in the theorem describes a property of every $n \times n$ singular matrix.

- For instance, an $n \times n$ singular matrix is not row equivalent to $I_n$, does not have $n$ pivot position, and has linearly dependent columns.
Example 1: Use the Invertible Matrix Theorem to decide if $A$ is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

Solution:

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$
So $A$ has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).

The Invertible Matrix Theorem *applies only to square matrices.*

For example, if the columns of a $4 \times 3$ matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $Ax = b$. 
Matrix multiplication corresponds to composition of linear transformations.

When a matrix $A$ is invertible, the equation $A^{-1}Ax = x$ can be viewed as a statement about linear transformations. See the following figure.

$A^{-1}$ transforms $Ax$ back to $x$. 
A linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is said to be **invertible** if there exists a function \( S : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
S(T(x)) = x \quad \text{for all} \quad x \in \mathbb{R}^n \quad (1)
\]
\[
T(S(x)) = x \quad \text{for all} \quad x \in \mathbb{R}^n \quad (2)
\]

**Theorem 9:** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear transformation and let \( A \) be the standard matrix for \( T \). Then \( T \) is invertible if and only if \( A \) is an invertible matrix. In that case, the linear transformation \( S \) given by \( S(x) = A^{-1}x \) is the unique function satisfying equation (1) and (2).
Proof: Suppose that $T$ is invertible.

Then (2) shows that $T$ is onto $\mathbb{R}^n$, for if $b$ is in $\mathbb{R}^n$ and $x = S(b)$, then $T(x) = T(S(b)) = b$, so each $b$ is in the range of $T$.

Thus $A$ is invertible, by the Invertible Matrix Theorem, statement (i).

Conversely, suppose that $A$ is invertible, and let $S(x) = A^{-1}x$. Then, $S$ is a linear transformation, and $S$ satisfies (1) and (2).

For instance, $S(T(x)) = S(Ax) = A^{-1}(Ax) = x$.

Thus, $T$ is invertible.