6.2 Orthogonal Sets

- Recall: two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if
  \[ \sum u_i v_i = \mathbf{u}^T \mathbf{v} = \mathbf{u} \cdot \mathbf{v} = 0. \]

- A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ from $\mathbb{R}^n$ is an orthogonal set if each pair of distinct vectors from that set is orthogonal, i.e.,
  \[ i \neq j \implies \mathbf{u}_i \cdot \mathbf{u}_j = 0. \]

- Examples:
  - The standard basis of $\mathbb{R}^n$.
  - The set $\{\mathbf{u}, \mathbf{0}\}$.
  - Example 1, textbook, the set
    \[ S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\} \]
• **Theorem 4**, p. 340, textbook. If

\[ S = \{u_1, u_2, \ldots, u_p\} \]

is an orthogonal set of **nonzero** vectors in \( \mathbb{R}^n \), then \( S \) is linearly independent. (Hence \( S \) is a basis of \( \text{span}(S) \).)

\[
Z = \sum_{i=1}^{p} \lambda_i u_i = 0 \quad \Rightarrow \quad \lambda_k = 0 \quad k = 1, \ldots, p
\]

\[
\langle u_k \rangle \cdot Z = u_k \cdot \sum_{i=1}^{p} \lambda_i u_i = \sum_{i=1}^{p} \lambda_i u_k \cdot u_i
\]

\[
= \sum_{i=1}^{p} \lambda_i u_k \cdot u_i \quad \Rightarrow \quad \lambda_k \neq 0
\]

\[
\lambda_k = \frac{u_k \cdot Z}{u_k \cdot u_k}
\]
Naturally, an **orthogonal basis** for a subspace $W$ of $\mathbb{R}^n$ is a basis for $W$ that is also an orthogonal set.

For example, the set in Example 1 is an orthogonal basis of $\mathbb{R}^3$.

Orthogonal Bases are nice! You can compute coefficients without solving a linear system.

Suppose

$$B = \{u_1, u_2, \ldots, u_p\}$$

is a basis of a subspace $W$ of $\mathbb{R}^n$,

$$B = [u_1, u_2, \ldots, u_p],$$

and $y$ is a vector in $W$. Then, in general, computing the coordinate vector

$$[y]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

of $y$ requires the solution of the linear system

$$B[y]_B = y.$$

However, if $B$ is an orthogonal basis we can compute the components of $[y]_B$ directly:

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}.$$
Example 2, p. 341. Express the vector

$$y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

as a linear combination of the vectors in the set

$$S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

in Example 1.

$$y = \sum_{i=1}^{3} u_i \cdot y u_i$$

$$= \frac{11}{11} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + \frac{-12}{6} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \frac{-33}{33/2} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 2 + 1 \\ 1 - 4 + 4 \\ 1 - 2 - 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$
Orthogonal Projections onto a Line

• Again, this is a review and generalization from Math 2210.

• Given a non-zero vector \( u \) in \( \mathbb{R}^n \) we wish to write \( y \) in \( \mathbb{R}^n \) as a multiple of \( u \) and a vector orthogonal to \( u \).

• That is we wish to write

\[
y = \hat{y} + z = \alpha u + z
\]

where

\[
z \cdot u = 0.
\]

• We want formulas for \( \alpha \) and \( z \). They are easy to obtain.

\[
(\hat{y} - \alpha u) \cdot u = 0
\]

\[
y \cdot u = \alpha u \cdot u
\]

\[
\alpha = \frac{y \cdot u}{u \cdot u}
\]

\[
z = y - \frac{y \cdot u}{u \cdot u} u
\]
• Example 3, pg. 342, textbook. Let

\[ y = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \]

Write \( y \) as a linear combination of a vector in \( \text{Span}\{u\} \) and a vector that is orthogonal to \( u \).

\[ \alpha = \frac{y \cdot u}{u \cdot u} = \frac{4 \cdot 2}{2 \cdot 2} = 2 \]

\[ \alpha u = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \]

\[ Z = y - \alpha u = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \]

**Figure 1.** Example 2.
• An orthogonal set is called an **orthonormal set** is all of its vectors are unit vectors.

• Example: The standard basis, and any (nonempty) subset of it.

• **Theorem 6**, p. 345. An $m \times n$ matrix $U$ has orthonormal columns if and only if

$$U^TU = I$$

(where $I$ is the $n \times n$ identity matrix.).
• **Theorem 7**, p. 345. Let $U$ be an $m \times n$ matrix with orthonormal columns, and let $x$ and $y$ be vectors in $\mathbb{R}^n$. Then:

a. $\|Ux\|^2 = \|x\|^2$

b. $(Ux) \cdot (Uy) = x \cdot y$

c. $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$

\[
\|Ux\|^2 = (Ux)^T Ux = x^T U^T U x = x^T x = \|x\|^2
\]

\[
(Ux) \cdot (Uy) = (Ux)^T (Uy) = x^T U^T U y = x^T y = x \cdot y
\]
The Pythagorean Theorem

• Suppose \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal. Then

\[
\| \mathbf{u} \pm \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2.
\]

• More precisely, we should say that

\[
\| \mathbf{u} \pm \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 \iff \mathbf{u} \cdot \mathbf{v} = 0.
\]

\[
\| \mathbf{u} - \mathbf{v} \|^2 = (\mathbf{u} - \mathbf{v})^T(\mathbf{u} - \mathbf{v})
\]
\[ \begin{align*}
&= u^T u - 2u^T v + v^T v \\
&= \|u\|^2 - 2u^T v + \|v\|^2
\end{align*} \]

\[ u, v \in \mathbb{R}^n \]

\((u, v) \in \mathbb{R} \text{ inner product if} \]

\((u, v) = (v, u) \]

\((u, u) > 0 \]

\((u, u) = 0 \Rightarrow u = 0 \]
\[(u + v, w) = (u, w) + (v, w)\]
\[(ku, v) = k(u, v)\]

**Ex.:** \((u, v) = u \cdot v\)

**Ex.:** \((u, v) = \sum_{i=1}^{n} w_i u_i v_i\), \(w_i > 0\)

**Ex.:** \((u, v) = u^T A v\)

\[A = A^T\]

\(A\) non-singular

\(u^T A u > 0\)

\[V = P_u\]

\((p, q) = \int_0^1 p(x) q(x) dx\)