6.1 Inner Product, Length, Orthogonality

• Today’s topic is familiar from Math 2210 where we discussed the dot product, the norm of a vector, and orthogonality of vectors.

• In our context, the terminology is slightly different, and we consider the space \( \mathbb{R}^n \) for general \( n \), instead of mostly, or just, \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).

• The inner product, previously called the dot product, of two vectors \( u \) and \( v \) in \( \mathbb{R}^n \), is defined to be

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= \mathbf{u}^T \mathbf{v} = [u_1 \quad u_2 \quad \ldots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= \sum_{i=1}^{n} u_i v_i.
\end{align*}
\]

• Examples:

\[
\begin{align*}
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= 2 \\
\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} &= 39
\end{align*}
\]
\( u^T v = \text{inner product} \)

\( uv^T = \text{outer product} \)

\[
\begin{bmatrix}
1 & 2 & 3 \\
6 & 7 & 8 \\
3 & 4 & 9 \\
9 & 0 & 27
\end{bmatrix}
\begin{bmatrix}
6 & 12 & 18 \\
3 & 6 & 9 \\
9 & 18 & 27
\end{bmatrix}
\]
• It’s straightforward to verify the following algebraic properties of the inner product:

• **Theorem 1, p. 333.** Let \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) be vectors in \( \mathbb{R}^n \), and \( c \) be a scalar. Then

  a. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
  
  b. \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \)
  
  c. \( (c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v}) \)
  
  d. \( \mathbf{u} \cdot \mathbf{u} \geq 0, \text{ and } \mathbf{u} \cdot \mathbf{u} = 0 \implies \mathbf{u} = \mathbf{0}. \)

• The **length** or **norm** \(^{-1}\) of a vector \( \mathbf{v} \) is defined by

\[
\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.
\]

• Examples.

\[
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\begin{bmatrix}
2 \\
1
\end{bmatrix}
= \sqrt{1+4+1} = \sqrt{6}.
\]

\(^{-1}\) also called **Standard Norm**, **Euclidean Norm**, or **2-norm**.
• If $\mathbf{u}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$ then $\|\mathbf{u}\|$ agrees with our ordinary concept of the length of a vector.
• Identifying points and vectors as usual, the distance between two vectors (points) $u$ and $v$ is given by $\|u - v\|$.
• The concept of orthogonality in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) generalized to orthogonality in \( \mathbb{R}^n \).

• Definition: Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal (or perpendicular) if

\[
\mathbf{u} \cdot \mathbf{v} = 0.
\]
• In 2210 we learned that

\[ \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta) \]  

(1)

where \( \theta \) is the angle formed by \( \mathbf{u} \) and \( \mathbf{v} \).

• This works also in \( \mathbb{R}^n \). You can take (1) as the definition of \( \theta \).

\[ \text{the zero vector is orthogonal to all vectors in } \mathbb{R}^n. \]

\[ \|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \]

\[ (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \]

\[ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta \]
Orthogonal Complements

• Suppose $W$ is a subspace of $\mathbb{R}^n$. Then the set

$$W^\perp = \{z : z \text{ is orthogonal to all vectors in } W\}$$

is a linear space, called the **orthogonal complement** of $W$.

• $W^\perp$ is read as "$W$-perpendicular" or, more commonly, just "$W$-perp".

• Example: line and plane in $\mathbb{R}^3$. 

![Diagram of line and plane orthogonal to each other]
\[ W^\perp \text{ is a linear space} \]
\[ W^\perp = \left\{ z : z \cdot w = 0 \quad w \in W \right\} \]
\[ z \cdot w = 0 \]
\[ y \cdot w = 0 \]
\[ (z + y) \cdot w = z \cdot w + y \cdot w = 0 + 0 = 0 \]
\[ (\alpha z) \cdot w = \alpha (z \cdot w) = \alpha \cdot 0 = 0 \]

\[ A \text{ m x n} \quad A = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix} = \begin{bmatrix} y_1^T \\ \vdots \\ y_m^T \end{bmatrix} \]

\[ \text{Null } A = \left\{ z : Az = 0 \right\} \]

\[ \text{Col } A = \text{span} \left\{ c_1, \ldots, c_n \right\} = \left\{ z : z = Ax \right\} \]

\[ \text{Row } (A) = \text{span} \left\{ r_1^T, \ldots, r_m^T \right\} \]
\[ = \left\{ z : z^T = y^T A \right\} \]
\[ = \text{Col } (A^T) \]

\[ \left( \text{Row } (A) \right)^\perp = \text{Null } (A) \]