Math 2270-6

Notes of 3/22/19

The Gershgorin Theorem

• There should be a prize for the mathematical theorem that maximizes the ratio of usefulness and notoriety.

• My vote for the most useful and least known mathematical fact would go to the Gershgorin Theorem.


• The basic idea underlying the Gershgorin Theorem is this: the eigenvalues of a diagonal matrix are the diagonal entries. If a matrix is close to being diagonal then its eigenvalues should be close to the diagonal entries.

• The Gershgorin Theorem makes this notion precise. It says:

**Gershgorin Theorem.** Suppose $A$ is an $n \times n$ matrix, and $\lambda$ is one of its eigenvalues. Then, for some $i \in \{1, 2, \ldots, n\}$

$$|a_{ii} - \lambda| \leq \sum_{j \neq i} |a_{ij}|.$$
• In other words, every eigenvalue lies in some
circle whose center is a diagonal entry of $A$, and whose radius equals the sum of the abso-
lute values of the off-diagonal entries in that row.

• Those circles are referred to as the Gersh-
gorin Circles.

• Before seeing why this is true and extending
this idea, let’s look at some examples.

• **Example 0.** Suppose $A$ is diagonal. Than
the radii of the Gershgorin circles are 0, and
the eigenvalues are the diagonal entries.

**Example 1.** Suppose

$$A = \begin{bmatrix}
-10 & 4 & 5 \\
3 & 10 & 5 \\
4 & 8 & 20
\end{bmatrix}.$$  

$$\begin{bmatrix}
-10 & 0 & 0 \\
0 & 10 & 0 \\
0 & 0 & 20
\end{bmatrix}$$

• The eigenvalues of $A$ (computed with matlab) are

$$\lambda_1 = 10.97, \quad \lambda_2 = 7.00 \quad \lambda_3 = 23.97$$

• The Gershgorin Circles and the eigenvalues
are shown in Figure 1.

• Note that since the origin is not contained in
any of the circles 0 is not an eigenvalue, which
implies that the matrix is non-singular.
• This observation generalizes. The matrix $A$ is \textbf{strictly diagonally dominant} if for all $i = 1, 2, \ldots, n$

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$
• It follows immediately from Gershgorin’s Theorem:

    A strictly diagonally dominant matrix is non-singular
It's easy to see that Gershgorin's Theorem is true.

\[ A\mathbf{x} = \lambda \mathbf{x} \quad \mathbf{x} \neq 0 \]

\[
\begin{bmatrix}
-3 \\
-2/3 \\
1
\end{bmatrix} \quad \begin{bmatrix}
1 \\
-2/3 \\
-1/3
\end{bmatrix}
\]

\[
\max_{j=1,\ldots,n} |x_j| = x_i = 1
\]

\[
\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i = \lambda
\]

\[
(\lambda - a_{ii})x_i = \lambda - a_{ii} = \sum_{j=1}^{n} a_{ij} x_j
\]

\[
= a_{ii} x_i + \ldots + a_{i-1, i} x_{i-1} + a_{i+1, i} x_{i+1} + \ldots + a_{in} x_n
\]

\[
|\lambda - a_{ii}| = \left| \sum_{j \neq i} a_{ij} x_j \right| \quad \left| -4 + 3 \right| \leq |\lambda - 4| + 3
\]

\[
\leq \sum_{j \neq i} |a_{ij} x_j| \quad \left| x_j \right| \leq 1
\]

\[
\leq \sum_{j \neq i} |a_{ij}|
\]

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The eigenvalues of $A$ are also the eigenvalues of $A^T$. Applying Gershgorin’s Theorem to the transpose of $A$ gives the result that the eigenvalues lie in the union of all circles with the same centers, but the radii being the sum of the absolute values of the off-diagonal entries in the corresponding column.
Revisit Example 1:

\[
A = \begin{bmatrix}
-10 & 4 & 5 \\
3 & 10 & 5 \\
4 & 8 & 20
\end{bmatrix}.
\]

Figure 2. Example 1 revisited.
It’s not true in general that every Gershgorin Circle contains an eigenvalue.

- Example: Suppose

\[
A = \begin{bmatrix} 1 & 1 \\ 16 & 1 \end{bmatrix}
\]

\[
|A - \lambda \mathbb{I}| = \begin{vmatrix} 1 - \lambda & 1 \\ 16 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 16 = 0
\]

\[
(1 - \lambda)^2 = 16
\]

\[
1 - \lambda = \pm 4
\]

\[
\lambda = 5 \quad \text{or} \quad \lambda = -3
\]
Example 2: Suppose

\[ A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \]

Gershgorin’s Theorem by itself does not imply that 1 is an eigenvalue.

However, it is! We can tell by observing that \( A - I \) is singular, or finding an eigenvector!
On the other hand, it is true that any union of $k$ Gershgorin Circles that does not overlap with any of the remaining Gershgorin Circles contains precisely $k$ eigenvalues, counting multiplicity.

- To see this let

\[ A = B + D \]

where $D$ is diagonal and the diagonal of $B$ is zero.

- For example, for the matrix in Example 1 we get

\[
A = \begin{bmatrix} -10 & 4 & 5 \\ 3 & 10 & 5 \\ 4 & 8 & 20 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ 3 & 0 & 5 \\ 4 & 8 & 0 \end{bmatrix} + \begin{bmatrix} -10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 20 \end{bmatrix}.
\]

- Next, let

\[ A(t) = B + tD. \]

Clearly,

\[ A(0) = D \quad \text{and} \quad A(1) = A. \]

The Gershgorin circles of $D$ are actually points (which are the eigenvalues of $D$). As $t$ changes from 0 to 1, the Gershgorin Circles of $A(t)$ start out as those of $D$ and grow until they
become those of $A(1)$. Since the centers don’t change, once they overlap for some $t = t_0$ they will overlap for all $t > t_0$. In particular, the Gershgorin Circles forming the set $S$ do not overlap with any other circles for all non-negative $t \leq 1$.

• Now, the eigenvalues of a matrix depend continuously on the matrix. Thus for no $t \in [0, 1]$ can an additional eigenvalue wander into $S$. or out of S
• We are done with chapter 5, eigenvalues and eigenvectors.

• Next: return to topics we have seen in Calc III:
  – dot products
  – distance
  – orthogonality
  – projections