Math 2270-6

Announcements

• no office hours today (after class).

• I changed the schedule for our discussion of chapter 4. There is a large overlap between chapters 2 and 4.

• There is a new syllabus online.

• Exam dates won’t change.
4.1 Vector Spaces and Subspaces

• Definition: A vector space is a nonempty set $V$ of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $u$, $v$, and $w$ in $V$, and for all scalars $c$ and $d$.

1. The sum of $u$ and $v$, denoted by $u+v$, is in $V$.
2. $u+v=v+u$.
3. $(u+v)+w=u+(v+w)$.
4. There is a zero vector $0$ in $V$ such that $u+0=u$.

5. For each $u$ in $V$, there is a vector $-u$ in $V$ such that $u+(-u)=0$.
6. The scalar multiple of $u$ by $c$, denoted by $cu$, is in $V$.
7. $c(u+v)=cu+cv$.
8. $(c+d)u=cu+du$.
9. $c(du)=(cd)u$.
10. $1u=u$.

- Also called a linear space
We only consider scalars that are real numbers. One can build a very similar theory of vector spaces based on the set of complex numbers. If you’ve studied algebra you are familiar with the concept of a field. (The sets of real numbers and complex numbers are special cases of fields.) One can develop a theory of linear spaces over any field.

- A **subspace** of a vector space \( V \) is a non-empty subset of \( V \) that is closed under addition and scalar multiplication.

Every subspace is a vector space itself.

- The primary examples of vector spaces are of course \( \mathbb{R}^n \) and subspaces of \( \mathbb{R}^n \). This includes, of course, the column and null spaces of a matrix.

- Of course! But there are more. Following are some examples. (As an exercise, check that the definition is satisfied.)

  - The column space of \( A^T \) (called the **row space** of \( A \)).
  
  - The null space of \( A^T \), i.e., the set of all \( x \) such that
    
    \[
    A^x = 0.
    \]

  - The set of all quadratic polynomials.
  - The set of all polynomials of degree \( n \)
• The set of all polynomials.

• The set of all real valued functions defined on some set (domain).

• The set of all functions that are continuous on \([a, b]\), usually denoted by \(C^0[a, b]\) or \(C[a, b]\).

• What do you think might be denoted by \(C^r[a, b]\)?

• The set of all functions that are square integrable on \(\mathbb{R}\):

\[
V = \left\{ f : \int_{-\infty}^{\infty} f^2(x) \, dx < \infty \right\}.
\]

• The set of all solutions of the differential equation

\[
y'' = k^2 y
\]

• The set of all \(m \times n\) matrices.

• The set of all upper triangular \(n \times n\) matrices.

• The set of all diagonal matrices.

• The set of all symmetric \(n \times n\) matrices (those that satisfy \(A = A^T\).)

• The set of all sequences

\[
x_0, x_1, x_2, x_3, \ldots
\]

• The set of all sequences \(x_0, x_1, x_2, \ldots\) that satisfy the infinitely many equations

\[
x_{n+2} - x_{n+1} - x_n = 0, \quad n = 0, 1, 2, \ldots
\]
• The set of all convergent sequences.
• The range of a linear transformation
• The null space of a linear transformation

Here are some examples of sets that are not vector spaces:

• A line or plane in \( \mathbb{R}^n \) not containing the origin.
• The set of all triangular matrices.
• The set of all non-singular (square) matrices
• The set of all singular (square) matrices.
• The set of all sequences \( x_0, x_1, x_2, \ldots \) that satisfy the infinitely many equations

\[
x_{n+2} - x_{n+1} - x_n = 1, \quad n = 0, 1, 2, \ldots
\]

• The set of all divergent sequences.
• The solution set of a linear system \( Ax = b \) (unless \( b = 0 \)).
Many concepts that we discussed previously for subspaces of $\mathbb{R}^n$ apply more or less directly.

- a **subspace** of a vector space $V$

- A **linear combination** of a (finite) set of vectors

- The **span** of a set of vectors

- A **spanning set** of a vector space

- **linear independence** of a set of vectors

- A **basis** of a vector space

- The **dimension** of a vector space.

let’s go back over the list of vector spaces and think about their dimension.
Some of the vector spaces we discussed today do not possess finite bases. They are infinite dimensional.

In a profound sense, all vector spaces of a given finite dimension are essentially similar. (The technical term is that they are isomorphic.)

In other words, there is only \( \mathbb{R}^n \).

- We need to look more closely into all these matters.