Math 2270-6

Notes of 2/15/2019

Announcements

• no class on Monday

• however, hw 7 will open on Monday as usual

Due to the recent snow day I had to shift Exams 3 and 4 by one week, so that they would take place after we finish the relevant chapters. There is a new syllabus online, and the revised remaining exam dates are

Exam 3: March 20 (after the break)
Exam 4: April 17

Mark your calendars.

Notes on Exam:

• 60 points correspond to 100 percent on the exam. The average grade on the Exam was B+, 17 people received an A.

• Unfortunately there was a typo in problem 5. I said $A_{12}$ is $q \times q$, rather than $A_{22}$ is $q \times q$. Luckily the problem still made sense, we just have to assume $p = q$. The answer

\[
\begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
p \\
q
\end{bmatrix}
\]
given in the answer sheet does not change. I graded the problem as if it was stated as intended, but as promised in the instructions, I gave extra credit (5 points) to everybody who queried the statement of the sizes of the block matrix. (As a consequence, a few people got more than 60 points on the exam.)

• We did problem 2 in class on the day before the exam. I was surprised (and impressed) by the number of people who came up with a way to write the matrix as the sum of two rank-1 matrices that differed from the way we did it in class.

• You can multiply with a matrix inverse from the left or the right, but you can’t divide by a matrix. It does not make sense to use fraction format for problem 5.

• I am worried by the comparatively low scores on the T/F question. Nobody got them all right. On the other hand, I believe understanding the concepts and connections is more important than computational skills. I am planning to increase the number of T/F questions on the remaining exams.
3.1 Determinants

- This is a very short chapter (3 sections).
- Determinants are numbers associated with **square matrices**.
- Every square matrix has a determinant, only square matrices have a determinant.
- For a square matrix $A$ that number is usually written as $\det A$ or $|A|$. Note that in this context the vertical bars do not mean absolute values!
- We’ll study determinants for two major reasons:
  1. A (square) matrix is invertible if and only if its determinant is non-zero.
  2. Determinants will be essential in studying *eigenvalues* in chapter 5.
- Let me also remind me of his highly popular video lectures on
  Determinants are introduced in Lecture 18.
• There is also a 764 page monograph: Thomas Muir, *A Treatise on the Theory of Determinants*, revised and enlarged by William Metzler, Dover, 1933.

• We’ll follow the textbook.

• Recall that we found the inverse of a $2 \times 2$ matrix to be

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}.
\]

• The number $a_{11}a_{22} - a_{12}a_{21}$ is defined to be the determinant of $A$.

• Clearly, a $2 \times 2$ matrix $A$ is invertible if and only if its determinant is zero.

• For $n > 2$ the determinant of $A$ is defined recursively. Suppose $A$ is an $n \times n$ matrix where $n > 1$. We define $A_{ij}$ to be the $(n-1) \times (n-1)$ matrix obtained from $A$ by removing the $i$-th row and the $j$-th column.
• Example:

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 17
\end{bmatrix}
\]

\[
A_{24} = \begin{bmatrix}
1 & 2 & 3 \\
9 & 10 & 11 \\
13 & 14 & 15
\end{bmatrix}
\]
• With this notation, the textbook defines the determinant as follows: For \( n \geq 2 \), the **determinant** of an \( n \times n \) matrix \( A = [a_{ij}] \) is

\[
\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n} \\
= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \tag{1}
\]

• Examples

\[
A = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 2 & -1 \\
3 & -1 & 1
\end{bmatrix}
\]

\[
|A| = \begin{bmatrix}
1 & 1 & 1 \\
3 & 1 & 1
\end{bmatrix} \quad \text{no!}
\]

\[
|A| = 1 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix}
\]

\[
= 1 \cdot 1 + 2 - 5 = -2
\]

\[
= -3 + 0 + 1 = -2
\]
A major fact about determinants is that they can be computed using any row or column.

- The corresponding formula is usually expressed in terms of cofactors:

\[ C_{ij} = (-1)^{i+j} \det A_{ij}. \]

- The factor \((-1)^{i+j}\)

creates the familiar checkerboard pattern

\[
\begin{bmatrix}
+ & - & + & - & \ldots \\
- & + & - & + & \ldots \\
+ & - & + & - & \ldots \\
- & + & - & + & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

- Theorem 1 in the textbook gives the more general formulas

\[
\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \tag{2}
\]

for any choice of \(i\) and

\[
\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \tag{3}
\]
for any choice of $j$.

- Of course knowing nothing of determinants it is not at all obvious that these formulas should give the same number for all choices of $i$ and $j$, and all (square) matrices $A$.

- quote from the textbook: “We omit the proof of [this] fundamental theorem to avoid a lengthy digression.”

- By comparison, Strang starts out by defining the determinant to be the uniquely defined function associating a number with any square matrix that has these properties:

1. The determinant of the identity matrix is 1.

2. If you interchange two rows of the matrix the determinant changes its sign.

3. The determinant is a function that is linear in each row separately. (In other words, if you think of the determinant as a function of a specific row, keeping everything else constant you get a linear function.)

- Everything, including the formulas listed in Theorem 1 flow from there, starting with the proof that there is a unique function with these properties.

- In the textbook, we start with formula (1), state Theorem 1 without proof, and then go from there, deriving in particular the three properties listed above.

- We’ll go along with that ...
• When using the cofactor formula we want to take advantage of zero entries.

• For example, compute the determinant of

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

\[
\det(A) = - \begin{vmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
\end{vmatrix}
\]

\[
= - \begin{vmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
\end{vmatrix}
\]

\[
= + \begin{vmatrix}
1 & 1 \\
0 & 0 \\
\end{vmatrix}
= -1
\]
• Theorem 2: The determinant of a triangular matrix equals the product of the diagonal entries.

• This follows easily from the cofactor expansion. We’ll just look at a simple example that will illustrate the pattern. Let $x$ denote arbitrary entries. Compute the determinant of

\[
A = \begin{bmatrix}
a & 0 & 0 & 0 \\
x & b & 0 & 0 \\
x & x & c & 0 \\
x & x & x & d
\end{bmatrix}
\]

\[
|A| = a \begin{vmatrix} b & 0 & 0 \\ x & c & 0 \\ x & x & d \end{vmatrix}
= a b \begin{vmatrix} c & 0 \\ x & d \end{vmatrix}
= a b c d
\]
- The cofactor expansion becomes very easy if there are many zero entries that are distributed in a beneficial pattern.

However, the cofactor expansion is prohibitively expensive in terms of computational effort if we cannot make use of the presence of zero entries.

- Assuming all entries are non-zero, computing the determinant of an $n \times n$ matrix requires the computation of

$$n \times (n - 1) \times (n - 2) \times 3 \times 2 \times 1 = n!$$

products.

- wikipedia has me acknowledge the source of the picture: Argonne National Laboratory’s Flickr page.

- Currently\(^{-1}\), the world’s fastest supercomputer (see Figure 1) can perform about $100 \times 10^{20}$ (a hundred quadrillion\(^{-2}\)) multiplications per second.

- How long does it take to compute the determinant of an $n \times n$ matrix by the cofactor expansion, on that computer?

\(^{-1}\) see https://en.wikipedia.org/wiki/Supercomputer
\(^{-2}\) That’s about 10 billion times as fast as a fast desktop computer, see https://www.popsci.com/intel-teraflop-chip
The following table lists the time $T$ required to compute the determinant by the cofactor formula for four values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.00024 seconds</td>
</tr>
<tr>
<td>25</td>
<td>26 minutes</td>
</tr>
<tr>
<td>30</td>
<td>840 years</td>
</tr>
<tr>
<td>35</td>
<td>32 billion years</td>
</tr>
</tbody>
</table>
By comparison, the age of the Universe is estimated to be 14 billion years. Computing the determinant of a $35 \times 35$ matrix would therefore take more than twice as long as the Universe has existed.

A $35 \times 35$ matrix is small by today’s standards.

On the other hand, computing the determinant of a $1000 \times 1000$ matrix takes less than a second in Matlab.

- Evidently Matlab does not use the cofactor expansion. There must be a better way!