Math 2270-1

Notes of 9/24/2019

Review

- $A$ is square, $n \times n$.
- The determinant of $A$ is given by

$$\det A = |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

where

$$C_{ij} = (-1)^{i+j} |A_{ij}|$$

and $A_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by removing the $i$-th row and the $j$-th column.

- $C_{ij}$ is the $(ij)$-cofactor and the formula is the cofactor expansion of the determinant.

$$|A| = \sum_{\sigma} \text{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

- The determinant is linear in each row and column
- In particular, multiplying a row with a scalar multiplies the determinant with that scalar.
• Interchanging two rows of $A$ changes the sign of the determinant.

• Adding a multiple of a row to another row does not change the determinant.

• $A$ is invertible if and only if $\det A \neq 0$.

• Transposing does not change the determinant:

$$\det A^T = \det A.$$ 

• The determinant is multiplicative:

$$|AB| = |A||B|$$
3.3 Cramer’s Rule etc.

- Cramer’s Rule (named after Gabriel Cramer, 1704–1752) is useful for some theoretical calculations. It’s not a competitive numerical method!

- Consider the linear system

\[
Ax = b
\]

where once again \( A \) is an invertible square \((n \times n)\) matrix.

- define

\[
A_i(b) = \begin{bmatrix} a_1 & \ldots & a_{i-1} & b & a_{i+1} & \ldots & a_n \end{bmatrix}
\]

- In other words, \( A_i(b) \) is the matrix formed by replacing the \( i \)-th column of \( A \) by \( b \).

- Cramer’s rule states that

\[
x_i = \frac{|A_i(b)|}{|A|}
\]
Examples:

\[
A = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 52 \\ 69 \end{bmatrix} = 6
\]

\[
A \mathbf{x} = \mathbf{b}
\]

\[
\mathbf{x}_1 = \frac{\begin{vmatrix} 52 & 7 \\ 69 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 7 \\ 2 & 9 \end{vmatrix}} = \frac{-15}{-5} = 3
\]

\[
\mathbf{x}_2 = \frac{\begin{vmatrix} 1 & 52 \\ 2 & 69 \end{vmatrix}}{-5} = \frac{-35}{-5} = 7
\]
• Proof of Cramer’s Rule:

\[
Ax = b \quad \Rightarrow \quad x_i = ?
\]

\[
\bar{I}_i(x) = \begin{bmatrix}
I_{-i} & 0 \\
0 & \ddots \\
0 & 0 & 1
\end{bmatrix}
\]

\[
|\bar{I}_i(x)| = x_i
\]

\[
A \bar{I}_i(x) = A_i(b)
\]

\[
|A| |\bar{I}_i(x)| = |A_i(b)|
\]

\[
x_i = \frac{|A_i(b)|}{|A|}
\]
A Formula for $A^{-1}$

- Remember our definition of cofactors:

  $$C_{ij} = (-1)^{i+j} |A_{ij}|$$

- Next remember that the $j$-th column of $A^{-1}$ is the solution of

  $$Ax_j = e_j$$

  where, as usual, $e_j$ is the $j$-th column of the identity matrix.

- By Cramer’s Rule, the $(i, j)$-entry $x_{ij}$ of $A^{-1}$ is

  $$x_{ij} = \frac{\det A_i(e_j)}{\det A}.$$  

- Finally, expand $A_i(e_j)$ by cofactors about the $i$-th column to see that

  $$x_{ij} = \frac{\det A_i(e_j)}{\det A} = \frac{C_{ji}}{\det A}.$$  

- We get the formula

  $$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$
The matrix
\[ \text{adj} A = \begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix} \]
is called the adjugate of \( A \).

Note that the adjugate is the transpose of the matrix of cofactors!

Aside: \( |C_{11}| = a_{11} \)

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad |A| = ad - bc = D \]

\[ C = \text{matrix of cofactors} \]

\[ C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \quad C^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

\[ A^{-1} = \frac{\begin{bmatrix} d & -5 \\ -c & a \end{bmatrix}}{ad - bc} \]
• Examples

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \quad c_1 = \begin{bmatrix} 8 \\ -20 \\ \text{etc.} \end{bmatrix} \]
Geometric Interpretation of the determinant

- Recall these three properties:

1. The determinant of the identity matrix is 1.
2. If you interchange two rows of the matrix the determinant changes it sign.
3. The determinant is a function that is linear in each row separately. (In other words, if you think of the determinant as a function of a specific row, keeping everything else constant you get a linear function.)

- Aside: Strang uses these three properties to define determinants.

- Ignoring the sign change, these properties define the area or volume of the rectangle (in $R^2$) or parallelepiped (in $R^3$) formed by the 2 or 3 columns of the given matrix.

- The corresponding object in higher dimensions is called a parallelootope.
\[ A = [u \ v] \]

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\]
• The volume $V$ of a parallelootope defined by the columns of $A$ is therefore given by

$$V = |\det A|$$

where the vertical bars in this case do denote the absolute values.

• Examples
Linear Transformations

- The textbook discusses in some detail that applying a linear transformation to a geometric object multiplies the volume of the object by the absolute value of the determinant of the linear transformation.

- You actually already saw this principle in action when discussing the change of variable formula in multivariable calculus.