Math 2270-6

Announcements

• Today, 11:50, JTB 120, study session
• tomorrow review of chpt 2.
• Notes for the review are now online!
• Wednesday, Exam 2
• Definition of subspace $H$: 0 is in $H$ implies that $H$ is not empty
• Undergraduate Colloquium. Every Wednesday, 12:55, LCB 225. More info on

http://www.math.utah.edu/undergrad/colloquia.php

This week

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A Fractional Introduction to Fractals

Abstract: How long is the coast of Great Britain? This question, although at first seemingly simple, has a peculiar answer when approached theoretically from different length scales due to the fractal-type nature of coastlines. In this talk we will introduce the notion of fractals, their unique mathematical properties, and their prevalence in nature, as well as view examples of such strange and fascinating objects.
2.9 Dimension and Rank

- Last section of chapter 2.
- We’ll use Example 3, textbook, as a running example

\[
A = \begin{bmatrix}
2 & 5 & -3 & -4 & 8 \\
0 & -3 & 2 & 5 & -7 \\
0 & -6 & 4 & 14 & -20 \\
0 & -9 & 6 & 5 & -6
\end{bmatrix}
\]

- Applying row operations (exercise) gives the row echelon form

\[
A \rightarrow \begin{bmatrix}
2 & 5 & -3 & -4 & 8 \\
0 & -3 & 2 & 5 & -7 \\
0 & 0 & 0 & 4 & -6 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

- More row operations (exercise) give the reduced row echelon form:

\[
A \rightarrow \begin{bmatrix}
1 & 0 & 1/6 & 0 & 17/12 \\
0 & 1 & -2/3 & 0 & -1/6 \\
0 & 0 & 0 & 1 & -3/2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
x_3 = 1, \quad x_5 = 0, \quad x_1 = -\frac{1}{6}, \quad x_2 = \frac{2}{3}, \quad x_3 = 0, \quad x_5 = 1, \quad x_4 = \frac{3}{2}, \quad x_2 = \frac{1}{6}, \quad x_1 = -\frac{17}{12}
\]

\[x = x_3 b_3 + x_5 b_5\]
• Recall these terms:
  – linear combination
  – span
  – spanning set
  – linear independence
  – linear dependence relations
  – basis
  – dimension: The number of vectors in a basis. It’s the same for all bases of a given subspace.

• Column space of an $m \times n$ matrix. $\text{COL}(A) = \{ y : y = Ax \}$

• Null space of an $m \times n$ matrix. $\text{nul}(A) = \{ x : Ax = 0 \}$

• major fact: elementary row operations do not change the null space or its dimension.

• elementary row operations do change the column space, but not the dimension of the column space, since they preserve linear dependence relations and spanning sets. Both can be expressed as linear equations and row reductions give equivalent linear systems.

• Definition: The dimension of the column space is the rank of $A$

• The pivot columns of $A$ form a basis of the column space of the reduced row echelon form.

• because row operations do not change dependence relations (which are solution of $Ax = 0$) the pivot columns of the original form a basis
of the column space of the original matrix.

- We saw on Friday that the dimension of the kernel is the number of free variables.

- Since the number of free variables plus the number of pivot columns equals the number $n$ of columns we get the result

\[
\text{rank} A + \dim \text{Nul} A = n.
\]

- The number of linearly independent rows is also the number of pivots.

- row rank equals column rank.

- Since they are equal we usually just use the word \textbf{rank} instead of \textbf{row rank} or \textbf{column rank}.

- But it’s a remarkable, and non-obvious fact that the number of linearly independent rows of a matrix equals the number of its linearly independent columns.

- For example, a $2 \times 1000$ matrix cannot have more than 2 linearly independent columns.

- Of course, we can see this independently of rank considerations. All those columns are in $\mathbb{R}^2$, which is 2-dimensional, and so we can’t have more than two linearly independent columns.
Continuation of Invertibility Theorem

- Recall Section 2.3, Invertibility theorem.

Theorem 8: Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given $A$, the statements are either all true or all false.

a. $A$ is an invertible matrix.

b. $A$ is row equivalent to the identity matrix.

c. $A$ has $n$ pivot positions.

d. The equation $Ax = 0$ has only the trivial solution.

e. The columns of $A$ form a linearly independent set.

f. The linear transformation $x \rightarrow Ax$ is one-to-one.

g. The equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation maps $\mathbb{R}^\text{b}$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix $C$ such that $CA = I$.

k. There is an $n \times n$ matrix $D$ such that $CA = I$.
   (Of course, the left and right inverses $C$ and $D$ are actually equal.)

l. $A^T$ is an invertible matrix.

- With our new concepts and language we can now add these equivalent statements:

m. $\text{Col} A = \mathbb{R}^n$. 
n. \( \dim \text{Col} A = n. \)

p. \( \text{rank} A = n. \)

q. \( \text{Nul} A = \{0\}. \)

r. \( \dim \text{Nul} A = 0. \)
• Here is a variation on the theme that for square linear systems existence is equivalent with uniqueness, i.e., linear independence.

• Theorem 15. Let \( H \) be a \( p \) dimensional subspace of \( \mathbb{R}^n \). Any linearly independent set of exactly \( p \) elements in \( H \) is a basis. Also any spanning set of \( p \) elements is a basis.

\[
\beta = \{ b_1, b_2, \ldots, b_p \}
\]

\( \beta \) span.

\( \implies \{ b_1, \ldots, b_{p-1} \} \) is a s.s.

\( \beta \) l.i. basis

\( \beta \) l.i. not a spanning set

\( V \) not in span \( \beta \)

\( \{ b_1, \ldots, b_p, v \} \)
Coordinate Systems

• Suppose \( \{b_1, \ldots, b_p\} \) is a basis for \( H \). Then \( x \) in \( H \) can be written uniquely as a linear combination

\[
x = \sum_{i=1}^{p} c_i b_i
\]

of the basis vectors.

• Suppose it can be written in two ways.

\[
x = \sum_{i=1}^{p} c_i b_i = \sum_{i=1}^{p} d_i b_i
\]

• Then

\[
0 = x - x = \sum_{i=1}^{p} c_i b_i - \sum_{i=1}^{p} d_i b_i = \sum_{i=1}^{p} (c_i - d_i) b_i.
\]

• Because of the linear independence of the basis vectors we have \( c_i - d_i = 0 \), i.e., \( c_i = d_i \).

• The \( c_i \) are the coordinates of \( x \) with respect to the basis \( \beta \).
Numerical Rank Determination

- We have assumed in all of our examples that we can use exact arithmetic.

- Computers (and calculators) represent most numbers only approximately.

- For example, $1/10$ cannot be expressed exactly on a binary computer.

- Computer Arithmetic, therefore, is only approximate.

- As we understand the subject at this stage, our standard approach to rank determination would be to compute the row echelon form of a matrix and then count the number of pivots.

- However, matrix entries that would be zero in exact arithmetic might not (and usually aren’t) zero in approximate arithmetic.

- Similar comments apply to other tasks like solving linear systems or finding a basis of a given subspace.

- This is the tip of a large ice-berg.
• True story about the first computer program I ever wrote. I had read about a theory that dinosaurs became extinct because of a recessive lethal gene and did not believe the conclusions of that article. Let $p$ be the probability that a given dinosaur had one copy of that gene. I wanted to compute what would happen to the population for the following values of $p$:

\[
0 \quad \frac{1}{10} \quad \frac{2}{10} \quad \frac{3}{10} \quad \frac{4}{10} \quad \frac{5}{10} \quad \frac{6}{10} \quad \frac{7}{10} \quad \frac{8}{10} \quad \frac{9}{10} \quad 1
\]

• I wrote a code like this:

\[
\begin{align*}
p &= 0.0 \\
\text{REPEAT:} \\
& \quad \text{compute and print likelihood of survival} \\
& \quad p = p + 0.1; \\
& \quad \text{if } p = 1.0 \text{ STOP;} \\
& \quad \text{go to REPEAT;}
\end{align*}
\]

I expected 11 lines of printed output. What do you think actually happened?