Math 2270-6

Announcements

- hw 6 due Tuesday night (2/12)
- Exam 3 on chapter 2 on Wednesday, 2/13, as planned.

Notes of 2/8/18

Subspaces of $\mathbb{R}^n$

- $\mathbb{R}^n$ is a vector space or linear space. The textbook has an upcoming precise definition in section 4.1.

- However, in the mean time we’ll discuss subspaces of $\mathbb{R}^n$.

- A subspace of $\mathbb{R}^n$ is any set $H$ in $\mathbb{R}^n$ that has these properties:

  1. For each $\mathbf{u}$ and $\mathbf{v}$ in $H$, $\mathbf{u} + \mathbf{v}$ is in $H$.

  2. For each $\mathbf{u}$ in $H$ and each scalar $c$ the vector $c\mathbf{u}$ is in $\mathbb{R}^n$.

- In other words, the subspace is closed under addition and scalar multiplication.

Property 2, by choosing $c = 0$, implies that every subspace contains the origin. The textbook lists this as an additional requirement (presumably for emphasis), but it’s not necessary.
Examples

• The set containing just the origin.

\{0\}
• Lines through the origin.
- Planes containing the origin
• For any set

\[ S = \{v_1, v_2, \ldots, v_p\} \]

the span of that set:

\[ H = \text{span}(S) \]

\[
\left(3v_1 + 2v_2\right) + \left(v_1 + 7v_3\right) = 4v_1 + 2v_2 + 7v_3
\]

\[ S \left(3v_1 + 2v_2\right) = 15v_1 + 10v_2 \]
• $\mathbb{R}^n$ itself

obviously
• The column space of a matrix.

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 1 \\
2 & 1 \\
\end{bmatrix}
\]

\[
\text{col } A = \left\{ \mathbf{v} : \mathbf{v} = x_1 \begin{bmatrix}1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix}2 \\ 1 \end{bmatrix} \right\}
\]

\[
= \left\{ \mathbf{v} : \mathbf{v} = A\mathbf{x} \right\}
\]
• The null space (or kernel) of a matrix.

\[
\text{null}(A) = \ker(A) = \{ x : Ax = 0 \}
\]

\[
Au = Av = 0
\]

\[
A(u+v) = Au + Av = 0 + 0 = 0
\]

\[
A(cu) = c(Au) = c \cdot 0 = 0
\]

\[
A \text{ m} \times \text{n}
\]

\[
\text{col}(A) \text{ subspace of } \mathbb{R}^m
\]

\[
\ker(A) \text{ subspace of } \mathbb{R}^n
\]
Basis of a subspace

- A spanning set of a subspace $H$ is a set
  \[ S = \{v_1, v_2, \ldots, v_p\} \]
  of vectors in $H$ such that
  \[ H = \text{span}(S). \]

- Major concept: A spanning set $S$ of a subspace $H$ is a **Basis** of $H$ if $S$ is linearly independent.

**Examples:**

- lines

![Diagram of a line through the origin with a vector $v$ and a set $\{v\}$]
• Planes

\{ v, w \}

\{ [0], [1] \}

standard basis
\( \mathbb{R}^n \)

\( \{ e_i \} \)

\[ e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \]

\[ x = \sum_{i=1}^{n} x_i e_i \]
- Column Space of a matrix

pivot columns not obvious yet
• Null Space of a matrix

Elementary row operations do not change the null space.

they do change the column space!

• Suppose we have reduced $A$ to reduced row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2 = 0$

free variables $x_2$, $x_4$

$x_2 = 1$ $x_4 = 0$ $x_1 = -2$ $x_3 = x_5 = 0$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_4 = 1$ $x_2 = 0$ $x_3 = -3$ $x_1 = x_5 = 0$

$$\begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$
\[ \begin{bmatrix} x_2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ -3x_4 \\ x_4 \\ 0 \end{bmatrix} \]
• **Major fact:** All Bases of a given subspace have the same number of vectors. That number is the **dimension** of the subspace. (We’ll follow exercises 27 and 28 in page 161 of the textbook.)

\[ \text{basis } \{b_1, b_2, \ldots, b_p\} \text{ of } \text{H subspace of } \mathbb{R}^n \]

**Claim:** any larger set is linearly dependent

\[ \{a_1, a_2, \ldots, a_q\} \quad q > p \]

\[ A = [a_1, a_2, \ldots, a_q] \quad n \times q \]

\[ B = [b_1, b_2, \ldots, b_p] \quad n \times p \]

\[ a_i = Bc_i \quad i = 1, \ldots, q \]

\[ C = [c_1, c_2, \ldots, c_q] \quad p \times q \]

\[ A = BC \quad p < q \]

\[ x \neq 0 \quad CX = 0 \]

\[ BCx = 0 \]

\[ A\vec{x} = 0 \quad \vec{x} \neq 0 \]