

$$0. \dots E-16 = 0$$

Math 2270-1

Notes of 11/04/19

- Today and tomorrow: review
- Wednesday: Exam 3 on Chapters 4 and 5



The following list is neither self contained nor is it complete. Rather, the individual items should stir your memory about facts, concepts, and connections. If some do not then it is a good idea to review the associated material!

Chapter 4: Vector Spaces

- **Definition:** A **vector space**⁻¹⁻ is a nonempty set V of objects, called vectors, on which are defined two operations, called **addition** and **multiplication by scalars (real numbers)**, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V , and for all scalars c and d .

1. The sum of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} + \mathbf{v}$, is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

⁻¹⁻ Also called a **linear space**

4. There is a zero vector $\mathbf{0}$ in V such that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}. \quad (1)$$

5. For each \mathbf{u} in V , there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

6. The scalar multiple of \mathbf{u} by c , denoted by $c\mathbf{u}$, is in V .

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.

10. $1\mathbf{u} = \mathbf{u}$.

- A **subspace** of a vector space V is a non-empty subset of V that is closed under addition and scalar multiplication.



every subspace is a vector space itself.

- Examples of vector spaces:
 - The primary examples of vector spaces are of course \mathbb{R}^n and subspaces of \mathbb{R}^n .
 - The column space of a matrix.
 - The null space of a matrix.
 - The column space of A^T (called the **row space** of A).
 - The null space of A^T , i.e., the set of all \mathbf{x} such that

$$A^T \mathbf{x} = \mathbf{0}. \quad (2)$$

- The set of all quadratic polynomials.
- The set of all polynomials of degree n
- The set of all polynomials.
- The set of all real valued functions defined on some set (domain).
- The set of all functions that are continuous on $[a, b]$, usually denoted by $C^0[a, b]$ or $C[a, b]$.
- The set of all functions that are square integrable on \mathbb{R} :

$$V = \left\{ f : \int_{-\infty}^{\infty} f^2(x) dx < \infty \right\}. \quad (3)$$

- The set of all solutions of the differential equation

$$y'' = k^2 y \quad (4)$$

- The set of all $m \times n$ matrices.
- The set of all upper triangular $n \times n$ matrices.
- The set of all diagonal matrices.
- The set of all symmetric $n \times n$ matrices (those that satisfy $A = A^T$.)
- The set of all sequences

$$x_0, x_1, x_2, x_3, \dots \quad (5)$$

- The set of all sequences x_0, x_1, x_2, \dots that satisfy the infinitely many equations

$$x_{n+2} - x_{n+1} - x_n = 0, \quad n = 0, 1, 2, \dots \quad (6)$$

- The set of all convergent sequences.
- The range of a linear transformation
- The null space of a linear transformation



Here are some examples of sets that are **not** vector spaces:

- A line or plane in \mathbb{R}^n not containing the origin.
- The set of all triangular matrices.
- The set of all non-singular (square) matrices
- The set of all singular (square) matrices.
- The set of all sequences x_0, x_1, x_2, \dots that satisfy the infinitely many equations

$$x_{n+2} - x_{n+1} - x_n = 1, \quad n = 0, 1, 2, \dots \quad (7)$$

$$|A+B| \neq |A|+|B|$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

- The set of all divergent sequences.
- The solution set of a linear system $A\mathbf{x} = \mathbf{b}$ (unless $\mathbf{b} = \mathbf{0}$).
- A **linear combination** of a (finite) set of vectors is obtained by multiplying each vector with some scalar and adding up the products.
- The **span** of a set of vectors is the set of all linear combinations of those vectors.
- A **spanning set of a vector space** is a subset of the vector space whose span is the space.
- **linear independence of a set of vectors** means that the only way to get the zero vector as a linear

combination of the vectors is by picking all coefficients equal to zero.

- A **basis of a vector space** is a linearly independent spanning set of the space.
- All bases of a specific vector space have the same number of elements. That number is the **dimension** of the vector space.
- We saw that this is true by showing that if \mathcal{B} is a basis with k elements than any set of k elements is linearly dependent. To do that we expressed every vector in the larger set in terms of the basis, and obtained a homogeneous rectangular matrix problem that was certain to have a non-trivial solution.
- Suppose

$$\mathcal{V} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\} \quad (8)$$

is a basis of a vector space V and

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{b}_i \quad (9)$$

is a vector in V . Then the vector

$$[\mathbf{x}]_{\mathcal{V}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad (10)$$

is the **coordinate vector of \mathbf{x} with respect to the basis \mathcal{V}** .

- Two vector spaces V and W are **isomorphic** if there is a linear transformation from V to W that is one-to-one and onto.
- Two isomorphic vector spaces have the same structure. Essentially they are the same. They differ only in notation or interpretation. As my linear algebra teacher said long ago, one space is painted green, the other is painted red.
- An isomorphism is invertible!
- Given a basis $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of V and a basis $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ of W an isomorphism C can be defined by

$$C\left(\sum_{i=1}^n \alpha_i \mathbf{v}_i\right) = \sum_{i=1}^n \alpha_i \mathbf{w}_i. \quad (11)$$

- This is equivalent to saying

$$C(\mathbf{v}_i) = \mathbf{w}_i \quad (12)$$

and requiring C to be linear.

- Another way to think about this is that you map a vector $\mathbf{v} \in V$ to a vector $\mathbf{w} \in W$ that has the same coefficient vector with respect to \mathcal{W} as \mathbf{v} has with respect to \mathcal{V} .
- Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

$$V = \{ \{x_n\}, x_{n+2} - x_{n+1} - x_n = 0 \}$$

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\begin{aligned} &\{1, 0, 1, 1, 2, \dots\} \\ &\{0, 1, 1, 2, \dots\} \end{aligned}$$

$$x_{n+2} = x_{n+1} + x_n$$

$$C \begin{bmatrix} a \\ b \end{bmatrix} = a, b, a+b, a+2b, 2a+3b, \dots$$



In particular, all n -dimensional vector spaces are isomorphic to \mathbb{R}^n .



Thus in a sense the only finite dimensional vector spaces are \mathbb{R}^n for $n = 1, 2, 3, \dots$

- We can convert between different bases of the same space. Suppose we have three bases of \mathbb{R}^n .

$$\begin{aligned} \mathcal{I} &= \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} \\ \mathcal{B} &= \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\} \\ \mathcal{C} &= \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \end{aligned} \quad (13)$$

- \mathcal{I} is the **standard basis**.
- As usual, we associate the matrices

$$\begin{aligned} B &= [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \dots \quad \mathbf{b}_n] \\ \text{and} \\ C &= [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_n] \end{aligned} \quad (14)$$

with the bases \mathcal{B} and \mathcal{C} .

- B and C are square and invertible.
- A vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (15)$$

can be expressed variously as

$$\mathbf{x} = [\mathbf{x}]_{\mathcal{I}} = B[\mathbf{x}]_{\mathcal{B}} = C[\mathbf{x}]_{\mathcal{C}} \quad (16)$$

- It follows that

$$[\mathbf{x}]_{\mathcal{B}} = B^{-1}\mathbf{x} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = C^{-1}\mathbf{x}. \quad (17)$$

- We can convert between the bases \mathcal{B} and \mathcal{C} by the formulas

$$[\mathbf{x}]_{\mathcal{B}} = B^{-1}C[\mathbf{x}]_{\mathcal{C}} \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = C^{-1}B[\mathbf{x}]_{\mathcal{B}}. \quad (18)$$

- Suppose A is an $m \times n$ matrix. It defines a linear transformation

$$\mathbf{y} = A\mathbf{x} \quad (19)$$

from \mathbb{R}^n to \mathbb{R}^m . Suppose we want to express **the same linear transform** in terms of a basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\} \quad (20)$$

of \mathbb{R}^n and a basis

$$\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m\} \quad (21)$$

of \mathbb{R}^m .

- In other words, we want to find a matrix T such that

$$[\mathbf{y}]_{\mathcal{C}} = T[\mathbf{x}]_{\mathcal{B}} \quad (22)$$

- The situation is illustrated in this “commuting diagram”:

$$\begin{array}{ccccc}
 & \mathbf{x} & \longrightarrow & A\mathbf{x} & \\
 B\mathbf{x} & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m & \mathbf{x} \\
 \uparrow & \uparrow & & \downarrow & \downarrow \\
 \mathbf{x} & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m & C^{-1}\mathbf{x} \\
 & \mathbf{x} & \longrightarrow & T\mathbf{x} &
 \end{array} \tag{23}$$

- start in the lower left corner. Move to the lower right corner either by going directly to the right, or in three steps by going up, right, and then down. We want T to be such that in either way we get to the same vector.

- Clearly,

$$T = C^{-1}AB. \tag{24}$$

- By the same token,

$$A = CTB^{-1}. \tag{25}$$

- check the dimensions.



In the special case that $m = n$ and $B = C$ we get that

$$T = B^{-1}AB. \tag{26}$$

- In this case A and B are said to be similar, and the formula (or the matrix B) is called a **similarity transform**.

6. Eigenvalues and Eigenvectors

- An **eigenvector** of a square $(n \times n)$ matrix A is a **non-zero** vector \mathbf{x} such that

$$A\mathbf{x} = \lambda\mathbf{x} \quad (27)$$

for some scalar λ . λ is called the **eigenvalue** of A corresponding to the eigenvector \mathbf{x} . \mathbf{x} is an eigenvector corresponding to the eigenvalue λ .

- The pair (λ, \mathbf{x}) is sometimes called an **eigenpair** of A .



Note than any non-zero scalar multiple of an eigenvector is also an eigenvector, with the same eigenvalue.



the main difference between linear systems and eigenvalue problems is that eigenvalue problems are **nonlinear**!

- More insight can be gained by writing

$$A\mathbf{x} = \lambda\mathbf{x} \quad (28)$$

as

$$A\mathbf{x} - \lambda\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}. \quad (29)$$

- Any eigenvector is a non-trivial solution of the homogeneous linear system

$$(A - \lambda I)\mathbf{x} = 0. \quad (30)$$

- Every eigenvector is in the nullspace of $A - \lambda I$.
- Every non-zero vector in the nullspace of $A - \lambda I$ is an eigenvector of A .
- A square homogeneous linear system has a non-trivial solution if and only if the coefficient matrix is singular.



thus λ is an eigenvalue of A if and only if $A - \lambda I$ is singular.



Upshot: we have one more characterization of singularity. A square matrix A is singular if and only if 0 is an eigenvalue of A . It is invertible if and only if all eigenvalues of A are non-zero.

- Suppose $\mathbf{x}_i, i = 1, \dots, m$ are eigenvectors corresponding to the same eigenvalue λ . Then any (non-zero) linear combination of the eigenvectors is also an eigenvector:

$$A \sum_{i=1}^m \alpha_i \mathbf{x}_i = \sum_{i=1}^n \alpha_i A \mathbf{x}_i = \sum_{i=1}^n \alpha_i \lambda \mathbf{x}_i = \lambda \sum_{i=1}^n \alpha_i \mathbf{x}_i. \quad (31)$$

- Thus, if we add the zero vector to the set of eigenvectors corresponding to a specific eigenvalue, that

set is a linear space, the nullspace of $A - \lambda I$. That space is also called the **eigenspace of A corresponding to λ** .

- Important example: The eigenvalues of a **triangular** matrix are the **diagonal entries**, because if A is triangular and λ is an eigenvalue then $A - \lambda I$ is a triangular matrix with at least one zero entry on the diagonal. It is thus singular.



Row operations do not preserve eigenvalues or eigenvectors!

- A matrix is singular if and only if its determinant is zero. Thus we get the key result:

$$\lambda \text{ is an e.v.} \iff \det(A - \lambda I) = 0. \quad (32)$$

- The equation

$$\det(A - \lambda I) = 0. \quad (33)$$

is the **characteristic equation** of A .

- The function $f(\lambda) = |A - \lambda I|$ is a polynomial of degree n with leading coefficient $(-1)^n$
- We can see this using a cofactor expansion or the formula

$$\det A = \sum_{\sigma} \text{sign}(\sigma) \prod_{i=1}^n a_{i\sigma_i} \quad (34)$$

where the sum goes over all $n!$ permutations of the set $\{1, 2, \dots, n\}$ and the large symbol π indicates the product of n factors, one from each row i , and the column σ_i .

- The polynomial

$$p(\lambda) = \det(A - \lambda I) \quad (35)$$

is the **characteristic polynomial** of A



The eigenvalues of A are the **roots** of the characteristic polynomial.

- This means
 1. There are precisely n of them, properly counting multiplicity.
 2. They may be repeated.
 3. They may be complex.
 4. If there are complex eigenvalues then they occur in conjugate complex pairs.
- The natural way to compute eigenvalues and eigenvectors by hand proceeds in two steps:
 1. Compute the characteristic polynomial and find its roots.
 2. For each eigenvalue λ find the nullspace of $A - \lambda I$.

- This works well only for small matrices with exactly known entries.
- However, the opposite process, computing roots of polynomials by computing the eigenvalues of a suitable matrix works very well.
- For every polynomial p of degree n with leading term $(-1)^n$ there exists a matrix A whose characteristic polynomial is p . Check:

$$\det \left(\begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \lambda I \right) = (-1)^n \left[\lambda^n - \sum_{j=0}^{n-1} \alpha_j \lambda^j \right].$$

companion matrix

(36)

- Central in eigenvalue calculations is the concept of **similarity**.
- **Definition:** Two matrices A and B are **similar** if there is a non-singular matrix P such that

$$B = P^{-1}AP. \quad (37)$$

- Similar matrices have the same eigenvalues, and their eigenvectors are related in a straightforward way. To see this suppose that

$$A\mathbf{x} = \lambda\mathbf{x} \quad (38)$$

and note that

$$B(P^{-1}\mathbf{x}) = P^{-1}APP^{-1}\mathbf{x} = P^{-1}\lambda\mathbf{x} = \lambda(P^{-1}\mathbf{x}). \quad (39)$$

- In other words, the eigenvectors of B are those of A , multiplied with P^{-1} .
- Another way to see that similar matrices have the same eigenvalues is to observe that their characteristic polynomials are the same. Using the multiplicative property of determinants and the fact that the determinant of the inverse is the reciprocal of the determinant of the original matrix we see

$$\begin{aligned} |B - \lambda I| &= |P^{-1}AP - \lambda P^{-1}IP| \\ &= |P^{-1}(A - \lambda I)P| \\ &= |P^{-1}||A - \lambda I||P| \\ &= |A - \lambda I| \end{aligned} \quad (40)$$

- **Definition:** A matrix is **diagonalizable** if it is similar to a diagonal matrix.
- In other words, A is diagonalizable if there exists a diagonal matrix D and a non-singular matrix P such that

$$D = P^{-1}AP. \quad (41)$$

- This equation can be rewritten as

$$AP = PD. \quad (42)$$

- Suppose

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n] \quad (43)$$

and

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad (44)$$

- Note that the equation for the i -th column in (42) is precisely the eigenvector equation

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i. \quad (45)$$



A matrix is diagonalizable if and only if it has n linearly independent eigenvectors. The similarity transform to diagonal form is the matrix of eigenvectors and the similar diagonal matrix has the eigenvalues along the diagonal.

- A matrix that is not diagonalizable is called **defective**.
- A matrix is not defective if and only if it has a set of n linearly independent eigenvectors.



Invertibility is unrelated to Diagonalizability.

singular invertible

$$\text{defective:} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (46)$$

$$\text{diagonalizable:} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- It is sometimes useful to be able to construct a matrix with given eigenvalues and eigenvectors. Note that

$$D = P^{-1}AP \quad (47)$$

is equivalent to

$$A = PDP^{-1}. \quad (48)$$

Suppose you want to construct a matrix A with given eigenvalues and given eigenvectors. Proceed as follows:

1. Collect the eigenvectors into the matrix P , and the eigenvalues into the matrix D , as before.
2. Compute P^{-1} .
3. Compute

$$A = PDP^{-1}. \quad (49)$$

- It is not always possible to diagonalize a matrix. However, for all matrices A there exists a similarity transform to its **Jordan Canonical Form**⁻²⁻ (named after Camille Jordan, 1838-1922).
- The JCF is a block diagonal matrix

$$P^{-1}AP = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{bmatrix} \quad (50)$$

where each diagonal block is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} \quad (51)$$

- Apart from reordering the diagonal blocks the JCF is unique.
- Each Jordan block J_i corresponds to one eigenvector with eigenvalue λ_i .
- A matrix is diagonalizable if and only if all of its Jordan blocks are 1×1 .

⁻²⁻ The textbook mentions the Jordan Canonical Form in a footnote on page 294.

- The **algebraic multiplicity** of an eigenvalue is its order as a root of the characteristic polynomial. Its **geometric multiplicity** is the dimension of its eigenspace.
- Here is an example. Suppose the Jordan form of a matrix is given by

$$J = \begin{bmatrix} 2 & 1 & . & . & . & . & . & . & . & . & . \\ . & 2 & . & . & . & . & . & . & . & . & . \\ . & . & 2 & . & . & . & . & . & . & . & . \\ . & . & . & 3 & 1 & . & . & . & . & . & . \\ . & . & . & . & 3 & 1 & . & . & . & . & . \\ . & . & . & . & . & 3 & . & . & . & . & . \\ . & . & . & . & . & . & 3 & . & . & . & . \\ . & . & . & . & . & . & . & 4 & . & . & . \\ . & . & . & . & . & . & . & . & 4 & . & . \\ . & . & . & . & . & . & . & . & . & 5 & . \end{bmatrix} \quad (52)$$

- Entries indicated by dots are zero.
- The characteristic polynomial of this matrix is

$$p(\lambda) = |J - \lambda I| = (2 - \lambda)^3(3 - \lambda)^4(4 - \lambda)^2(5 - \lambda). \quad (53)$$

The number 2 is an eigenvalue of algebraic multiplicity 3 and geometric multiplicity 2, 3 is an eigenvalue of algebraic multiplicity 3 and geometric multiplicity 2, 4 is an eigenvalue of algebraic and geometric multiplicity 2, and 5 is an eigenvalue of algebraic and geometric multiplicity 1. The dimension of the space spanned by all eigenvectors

is the sum of the geometric multiplicities which is 7. The matrix is defective.

- A set of eigenvectors corresponding to distinct eigenvalues is linearly independent. (The word “distinct” means that no two of the eigenvalues are equal.)
- Recall that a matrix is diagonalizable if it has a set of n linearly independent eigenvectors.
- Thus a matrix with distinct eigenvalues is diagonalizable.



This implies, for example, that the JCF can be computed only in exact arithmetic.

- A non-diagonalizable matrix must have multiple eigenvalues.



The most important thing to know about complex eigenvalues is that **symmetric real matrices don't have any!** The textbook addresses this issue in problem 24 on page 303 (and later in chapter 7).

- But the argument is quite simple.
- For any matrix A or vector \mathbf{x} let

$$A^H = \bar{A}^T \quad \text{and} \quad \mathbf{x}^H = \bar{\mathbf{x}}^T \quad (54)$$

where the bar denotes conjugate complex.

- A complex matrix A is **Hermitian**⁻³⁻ if

$$A = \bar{A}^T. \quad (55)$$

- We will show that the eigenvalues of a Hermitian matrix are real.



Note that symmetric real matrices are special cases of Hermitian matrices.

- Suppose

$$A\mathbf{x} = \lambda\mathbf{x} \quad (56)$$

where $A = A^H$, and A , λ , and \mathbf{x} are all possibly complex. Taking the conjugate complex on both sides turns this into

$$\mathbf{x}^H A^H = \bar{\lambda} \mathbf{x}^H. \quad (57)$$

Left multiplying with \mathbf{x}^H in (56) and right multiplying with \mathbf{x} in (57) gives

$$\mathbf{x}^H A \mathbf{x} = \lambda \mathbf{x}^H \mathbf{x} \quad \text{and} \quad \mathbf{x}^H A \mathbf{x} = \bar{\lambda} \mathbf{x}^H \mathbf{x}. \quad (58)$$

Thus

$$\lambda \mathbf{x}^H \mathbf{x} = \bar{\lambda} \mathbf{x}^H \mathbf{x}. \quad (59)$$

This implies that $\lambda = \bar{\lambda}$, i.e., λ is real.

⁻³⁻ named after Charles Hermite, 1822–1901.

- **Gershgorin Theorem.** Suppose A is an $n \times n$ matrix, and λ is one of its eigenvalues. Then, for some $i \in \{1, 2, \dots, n\}$

$$|a_{ii} - \lambda| \leq \sum_{j \neq i} |a_{ij}|. \quad (60)$$

- In other words, every eigenvalue lies in some circle whose center is a diagonal entry of A , and whose radius equals the sum of the absolute values of the off-diagonal entries in that row.
- Those circles are referred to as the **Gershgorin Circles**.
- To see this suppose \mathbf{x} is an eigenvector of the $n \times n$ matrix A , with corresponding eigenvalue λ . Thus

$$A\mathbf{x} = \lambda\mathbf{x}. \quad (61)$$

- Since an eigenvector is determined only up to a non-zero factor we may assume that \mathbf{x} is normalized such that

$$\max_{j=1, \dots, n} |x_j| = x_i = 1 \quad (62)$$

for some i in $\{1, 2, \dots, n\}$. This fixes i . If there are several such indices i we pick any particular one of them.

- The i -th component of the vector equation (62) is

$$\sum_{j=1}^n a_{ij}x_j = \lambda x_i = \lambda. \quad (63)$$

- Subtracting $a_{ii}x_i = a_{ii}$ on both sides gives the equation

$$\lambda - a_{ii} = \sum_{j \neq i} a_{ij}x_j \quad (64)$$

- Taking absolute values on both sides, applying the triangle inequality, and observing that $|x_j| \leq 1$ for all j shows that λ lies in the Gershgorin Circle centered at x_i :

$$\begin{aligned} |\lambda - a_{ii}| &= \left| \sum_{j \neq i} a_{ij}x_j \right| \\ &\leq \sum_{j \neq i} |a_{ij}x_j| \\ &= \sum_{j \neq i} |a_{ij}||x_j| \\ &\leq \sum_{j \neq i} |a_{ij}| \end{aligned} \quad (65)$$



It's not true in general that every Gershgorin Circle contains an eigenvalue.



On the other hand, it is true that any union of k Gershgorin Circles that does not overlap with any of the remaining Gershgorin Circles contains precisely k eigenvalues, counting multiplicity.

Q & A 11/5/19

$$A = \begin{bmatrix} 7 & 0 & 13 \\ 9 & 3 & 23 \\ -5 & 0 & -7 \end{bmatrix}$$

$$\begin{aligned} & (7-\lambda)(-7-\lambda) \\ &= -(7-\lambda)(7+\lambda) \\ &= -7^2 + \lambda^2 \end{aligned}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 7-\lambda & 0 & 13 \\ 9 & 3-\lambda & 23 \\ -5 & 0 & -7-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 7-\lambda & 13 \\ -5 & -7-\lambda \end{vmatrix} \\ &= (3-\lambda)(\lambda^2 - 7^2 + 65) \\ &= (3-\lambda)(\lambda^2 + 16) \\ &\lambda = 3 \quad \lambda = \pm 4i \end{aligned}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ eigenvalues real if $|A| < 0$?

$$|A| = ad - bc$$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc = 0$$

Quadratic Formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda \text{ real if } (a+d)^2 > 4|A|$$

$$|A| < \frac{(a+d)^2}{4}$$

$$x \neq 0$$

$$A = \begin{bmatrix} -10 & 4 & 5 \\ 3 & 10 & -5 \\ -4 & 8 & 20 \end{bmatrix}$$

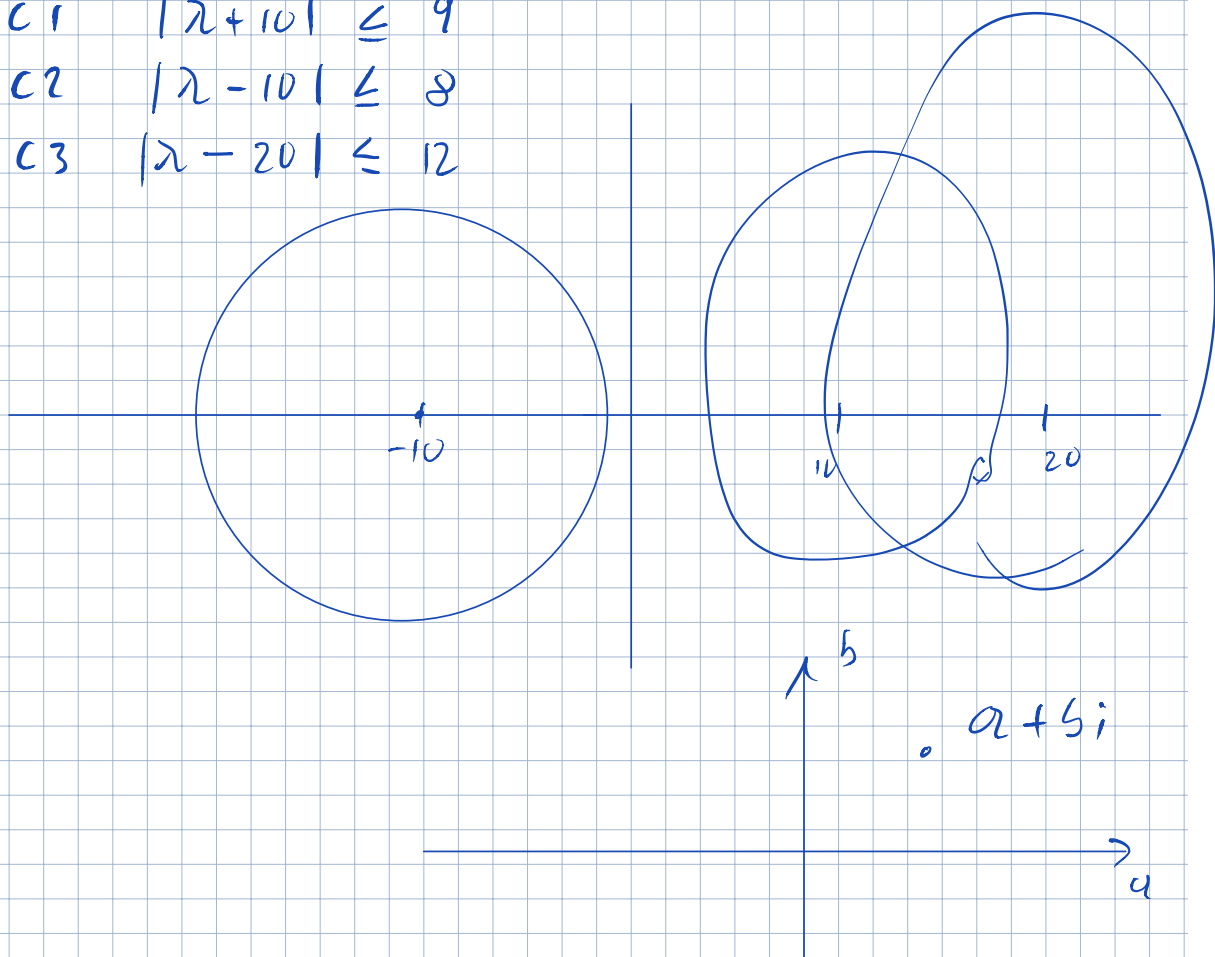
$$Ax = \lambda x \Rightarrow |\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$$

for some i

$$C1 \quad |\lambda + 10| \leq 9$$

$$C2 \quad |\lambda - 10| \leq 8$$

$$C3 \quad |\lambda - 20| \leq 12$$



5 ctd

$$D = ad - bc$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4D}}{2}$$

P_3 space of polyn of degree 3

$$\dim P_3 = 4$$

$$\{1, x, x^2, x^3\}^G$$

$$M_2 = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = [A]_B$$

$$G'(x^3 + \beta x^2 + \gamma x + \delta) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$\begin{bmatrix} \delta \\ \gamma \\ \beta \\ \alpha \end{bmatrix} = [p]_G$$

$$\alpha = a$$

$$\beta = b$$

$$\gamma = c$$

$$\delta = d$$

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \rightarrow \begin{bmatrix} \delta \\ \gamma \\ \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$

A is diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$$

$\Rightarrow A$ invertible

$$Ax = \lambda x$$

$$\max_{j=1, \dots, n} |x_j| = \underline{\underline{1}} = x_i$$

i -th eqn:

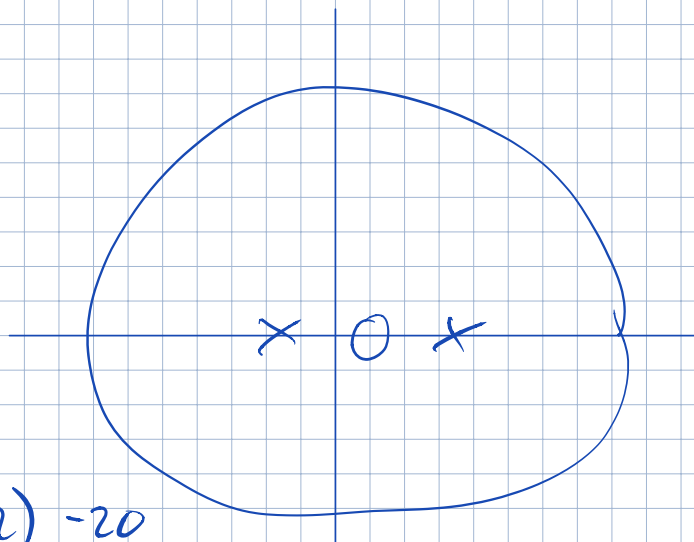
$$\sum_{j=1}^n a_{ij} x_j = \lambda x_i$$

$$| - a_{ii} x_i \quad \longleftrightarrow$$

$$\left(\begin{array}{l} \lambda x_i - a_{ii} x_i = \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j \\ = \lambda - a_{ii} \end{array} \right.$$

$$\begin{aligned} |\lambda - a_{ii}| &= \left| \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j \right| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij} x_j| \\ &\leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| |x_j| \\ &\leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \end{aligned}$$

$$\begin{bmatrix} 1 & 20 \\ 1 & 2 \end{bmatrix}$$



$$\begin{aligned} p(\lambda) &= (1-\lambda)(2-\lambda) - 20 \\ &= \lambda^2 - 3\lambda - 18 \\ &= (\lambda-6)(\lambda+3) \end{aligned}$$

Every union of k GCs not overlapping with any other GCs contains precisely k eigenvalues

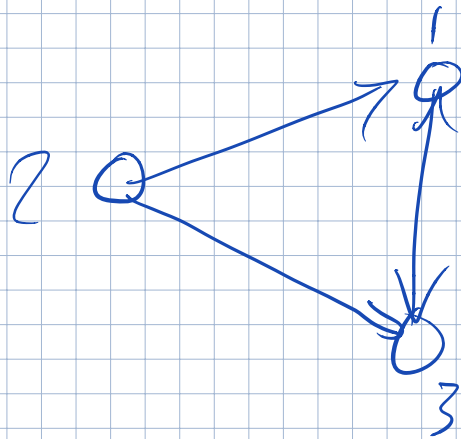
$$A(t) = D + tB$$

$$D = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} 0 & & a_{1j} \\ & \ddots & \\ a_{ij} & & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$$

$$A(0) = D$$

$$A(1) = A$$



$$A = \begin{bmatrix} 0 & 1/2 & 1 \\ 0 & 0 & 0 \\ 1 & 1/2 & 0 \end{bmatrix}$$

$$B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$G = (1-\epsilon)A + \epsilon B$$

$$\epsilon = \frac{1}{6}$$

$$= \frac{5}{6}A + \frac{1}{6}B$$

$$= \frac{5}{6} \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{bmatrix} + \frac{1}{18} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$Ge = e$$

$$e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(G - I)e = 0$$