

Math 2270-1

Notes of 9/25/2019

- Today's subject: generalize the notion of a vector space.
- General procedure:
 1. Start with a notion or a specific example.
 2. Come up with a precise definition.
 3. Investigate what objects satisfy the definition, and what properties are implied by the definition.
- In the past we applied the same idea in Calculus, to
 - limits
 - derivatives
 - integrals
- The definition often consists of a list of required properties, sometimes called **axioms**. That's the approach we'll take today.

- We start with \mathbb{R}^n . What key properties does \mathbb{R}^n have?

4.1 Vector Spaces and Subspaces

- **Definition:** (p. 192, textbook): A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called **addition** and **multiplication by scalars (real numbers)**, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V , and for all scalars c and d .

1. The sum of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} + \mathbf{v}$, is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a zero vector $\mathbf{0}$ in V such that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}.$$

5. For each \mathbf{u} in V , there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of \mathbf{u} by c , denoted by $c\mathbf{u}$, is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

- Exercises: Using only the axioms, prove that $\mathbf{0}$ is unique, $-\mathbf{u} = (-1)\mathbf{u}$, $0\mathbf{u} = \mathbf{0}$, and $c\mathbf{0} = \mathbf{0}$.



A phrase that is synonymous with “vector space” is **linear space**. The textbook does not use that phrase, but I believe it is used more commonly than “vector space”.



We only consider scalars that are real numbers. One can build a very similar theory of vector spaces based on the set of complex numbers.



If you’ve studied algebra you are familiar with the concept of a **field**. (The sets of real numbers and complex numbers are special cases of fields.) One can develop a theory of linear spaces over any field.

Examples

- \mathbb{R}^n
- Arrows in the plane or 3 dimensional space.
(Of course we then identify the arrows with
vectors in \mathbb{R}^2 or \mathbb{R}^3 .)

Sequences

Polynomials

Functions

- A **subspace** of a vector space V is a non-empty subset of V that is closed under addition and scalar multiplication.
- The textbook requires three properties (see p. 195).
- Definition: A **subspace** of a vector space V is a subset H of V that has three properties:
 - a. The zero vector of V is in H .
 - b. H is closed under vector addition. That is if \mathbf{u} and \mathbf{v} are in H , then $\mathbf{u} + \mathbf{v}$ are in H .
 - c. H is closed under scalar multiplication. That is if \mathbf{u} is in H and c is a scalar, then $c\mathbf{u}$ is in H .
- If H is non empty, and closed under addition and scalar multiplication, then

$$\mathbf{0} = \mathbf{v} + (-1)\mathbf{v}$$

is in H .

- Thus the only reason to have the requirement
 - a. is to make sure H is non-empty.



every subspace is a vector space itself.



An equivalent (exercise!) definition of “subspace” is: A subspace of a vectors space V is a subset H that is itself a vector space.

- For every vector space V , the set $\{\mathbf{0}\}$ is a subspace, called the **zero subspace**.

- Find examples of the previous examples of vector spaces.

Span

- the span of a set of vectors is a subspace.