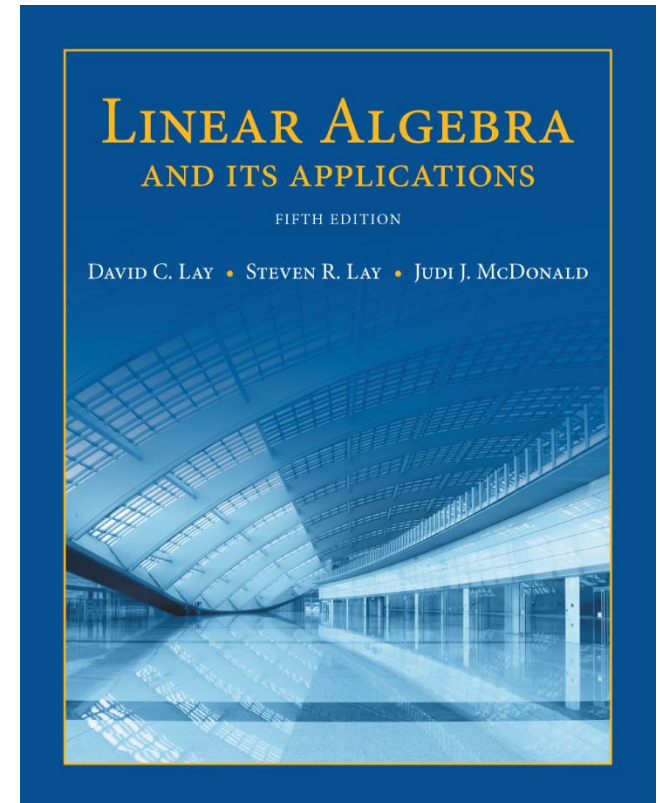


2

Matrix Algebra

2.1

MATRIX OPERATIONS



MATRIX OPERATIONS

- If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the i th row and j th column of A is denoted by a_{ij} and is called the (i, j) -entry of A . See the Fig. 1 below.
- Each column of A is a list of m real numbers, which identifies a vector in \mathbb{R}^m .

The diagram shows a matrix A with m rows and n columns. The entry a_{ij} is highlighted in a blue square. The entire row i is highlighted in light blue, and the entire column j is highlighted in light blue. The matrix is enclosed in large square brackets, with an equals sign and the letter A to its right. Above the matrix, the word "Column" is written in blue, with a blue j below it. To the left of the matrix, the words "Row i " are written in blue. Below the matrix, three blue arrows point upwards to the first, j th, and n th columns, labeled \mathbf{a}_1 , \mathbf{a}_j , and \mathbf{a}_n respectively in blue.

$$\begin{array}{c} \text{Column } j \\ \left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \text{Row } i & a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array} \right] = A \\ \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array} \end{array}$$

Matrix notation.

MATRIX OPERATIONS

- The columns are denoted by $\mathbf{a}_1, \dots, \mathbf{a}_n$, and the matrix A is written as

$$A = [a_1 \ a_2 \ \dots \ a_n]$$

- The number a_{ij} is the i th entry (from the top) of the j th column vector \mathbf{a}_j .
- The **diagonal entries** in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \dots$, and they form the **main diagonal** of A .
- A **diagonal matrix** is a square $n \times m$ matrix whose nondiagonal entries are zero.
- An example is the $n \times n$ identity matrix, I_n .

SUMS AND SCALAR MULTIPLES

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0 .
- The two matrices are **equal** if they have the same size (*i.e.*, the same number of rows and the same number of columns) and if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal.
- If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B .

SUMS AND SCALAR MULTIPLES

- Since vector addition of the columns is done entrywise, each entry in $A + B$ is the sum of the corresponding entries in A and B .
- The sum $A + B$ is defined only when A and B are the same size.
- **Example 1:** Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$,

and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Find $A + B$ and $A + C$.

SUMS AND SCALAR MULTIPLES

- **Solution:** $A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$ but $A + C$ is not defined because A and C have different sizes.
- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A .
- **Theorem 1:** Let A , B , and C be matrices of the same size, and let r and s be scalars.
 - a. $A + B = B + A$

SUMS AND SCALAR MULTIPLES

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

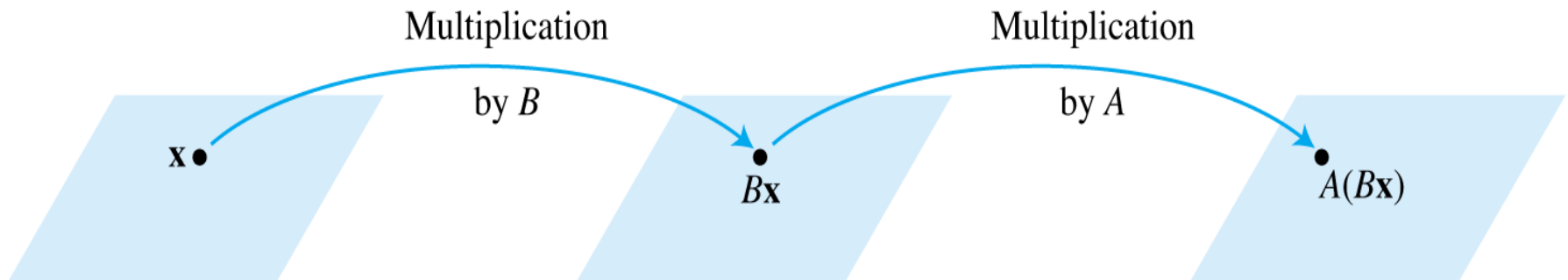
e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

- Each quantity in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal.

MATRIX MULTIPLICATION

- When a matrix B multiplies a vector \mathbf{x} , it transforms \mathbf{x} into the vector $B\mathbf{x}$.
- If this vector is then multiplied in turn by a matrix A , the resulting vector is $A(B\mathbf{x})$. See the Fig. 2 below.

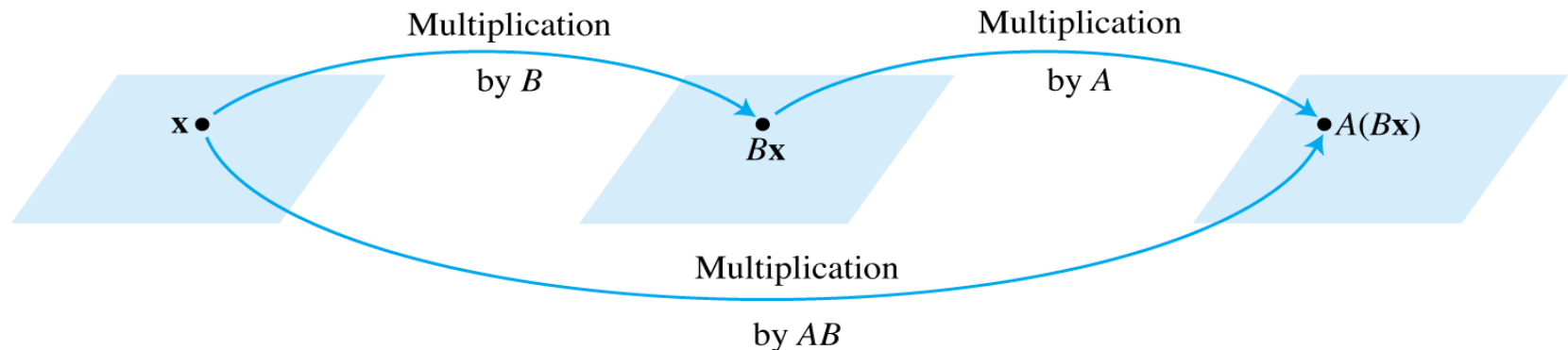


Multiplication by B and then A .

- Thus $A(B\mathbf{x})$ is produced from \mathbf{x} by a *composition of mappings*—the linear transformations.

MATRIX MULTIPLICATION

- Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB , so that $A(B\mathbf{x}) = (AB)\mathbf{x}$. See Fig. 3 below



Multiplication by AB .

- If A is $m \times n$, B is $n \times p$, and \mathbf{x} is in \mathbb{R}^p , denote the columns of B by $\mathbf{b}_1, \dots, \mathbf{b}_p$ and the entries in \mathbf{x} by x_1, \dots, x_p .

MATRIX MULTIPLICATION

- Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \dots + x_p\mathbf{b}_p$$

- By the linearity of multiplication by A ,

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1) + \dots + A(x_p\mathbf{b}_p) \\ &= x_1A\mathbf{b}_1 + \dots + x_pA\mathbf{b}_p \end{aligned}$$

- The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1, \dots, A\mathbf{b}_p$, using the entries in \mathbf{x} as weights.
- In matrix notation, this linear combination is written as

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

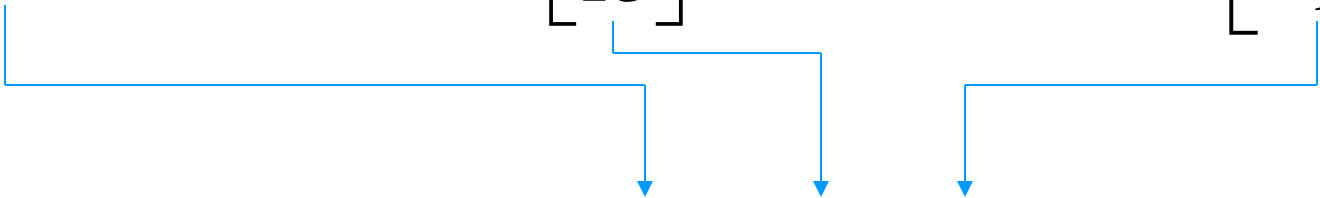
MATRIX MULTIPLICATION

- Thus multiplication by $\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$ transforms \mathbf{x} into $A(B\mathbf{x})$.
- **Definition:** If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \dots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \dots, A\mathbf{b}_p$.
- That is,
$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$
- *Multiplication of matrices corresponds to composition of linear transformations.*

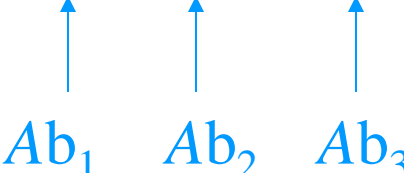
MATRIX MULTIPLICATION

- **Example 3:** Compute AB , where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.
- **Solution:** Write $B = [b_1 \quad b_2 \quad b_3]$, and compute:

MATRIX MULTIPLICATION

$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}, Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \quad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \quad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$


■ Then

$$AB = A[b_1 \quad b_2 \quad b_3] = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$


Ab_1 Ab_2 Ab_3

MATRIX MULTIPLICATION

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B .

Row—column rule for computing AB

- If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B . If $(AB)_{ij}$ denotes the (i, j) -entry in AB , and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

PROPERTIES OF MATRIX MULTIPLICATION

- **Theorem 2:** Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.
 - a. $A(BC) = (AB)C$ (associative law of multiplication)
 - b. $A(B + C) = AB + AC$ (left distributive law)
 - c. $(B + C)A = BA + CA$ (right distributive law)
 - d. $r(AB) = (rA)B = A(rB)$ for any scalar r
 - e. $I_m A = A = A I_n$ (identity for matrix multiplication)

PROPERTIES OF MATRIX MULTIPLICATION

- **Proof:** Property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known that the composition of functions is associative. Let

$$C = \begin{bmatrix} c_1 & \cdots & c_p \end{bmatrix}$$

- By the definition of matrix multiplication,

$$BC = \begin{bmatrix} Bc_1 & \cdots & Bc_p \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} A(Bc_1) & \cdots & A(Bc_p) \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

- The definition of AB makes $A(B\mathbf{x}) = (AB)\mathbf{x}$ for all \mathbf{x} , so

$$A(BC) = \begin{bmatrix} (AB)c_1 & \cdots & (AB)c_p \end{bmatrix} = (AB)C$$

- The left-to-right order in products is critical because AB and BA are usually not the same.
- Because the columns of AB are linear combinations of the columns of A , whereas the columns of BA are constructed from the columns of B .
- The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A .

PROPERTIES OF MATRIX MULTIPLICATION

- If $AB = BA$, we say that A and B **commute** with one another.

- **Warnings:**
 1. In general, $AB \neq BA$.
 2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
 3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

POWERS OF A MATRIX

- If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A :

$$A^k = \underbrace{A \cdots A}_k$$

- If A is nonzero and if \mathbf{x} is in \mathbb{R}^n , then $A^k \mathbf{x}$ is the result of left-multiplying \mathbf{x} by A repeatedly k times.
- If $k = 0$, then $A^0 \mathbf{x}$ should be \mathbf{x} itself.
- Thus A^0 is interpreted as the identity matrix.

THE TRANSPOSE OF A MATRIX

- Given an $m \times n$ matrix A , the **transpose** of A is the $n \times m$ matrix, denoted by A^T , whose columns are formed from the corresponding rows of A .

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r , $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

THE TRANSPOSE OF A MATRIX

- The transpose of a product of matrices equals the product of their transposes in the *reverse* order.