

Math 2270-1

Notes of 11/11/19

- Quick review: Two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n are **orthogonal** if

$$\mathbf{u} \bullet \mathbf{v} = 0.$$

They are **orthonormal** if they are orthogonal, and are also unit vectors:

$$\|\mathbf{u}\| = \|\mathbf{v}\| = 1.$$

An **orthogonal set**

$$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$$

is a set whose vectors are pairwise orthogonal, i.e.,

$$i \neq j \implies \mathbf{u}_i \bullet \mathbf{u}_j = 0.$$

An orthogonal set is an **orthonormal set** if it is orthogonal and all of its vectors are unit vectors. An orthogonal (**orthonormal**) **basis** is a basis that is also an **orthogonal (orthonormal)** set.

- Orthogonal Bases are nice. Suppose

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$$

is an orthogonal basis of a subspace W of \mathbb{R}^n and \mathbf{y} is in \mathbb{R}^n . Then

$$\hat{\mathbf{y}} = \sum_{i=1}^p \frac{\mathbf{y} \bullet \mathbf{u}_i}{\mathbf{u}_i \bullet \mathbf{u}_i} \mathbf{u}_i \quad (1)$$

is the **orthogonal projection of \mathbf{y} onto W** .
 In the special case that \mathbf{y} is in W then of course $\mathbf{y} = \hat{\mathbf{y}}$:

$$\mathbf{y} = \sum_{i=1}^n \frac{\mathbf{y} \bullet \mathbf{u}_i}{\mathbf{u}_i \bullet \mathbf{u}_i} \mathbf{u}_i \quad (2)$$

- In the special case that B is orthonormal the formulas (1) and (2) simplify to

$$\hat{\mathbf{y}} = \sum_{i=1}^p (\mathbf{y} \bullet \mathbf{u}_i) \mathbf{u}_i. \quad (3)$$

and

$$\mathbf{y} = \sum_{i=1}^n (\mathbf{y} \bullet \mathbf{u}_i) \mathbf{u}_i, \quad (4)$$

respectively.

- Any vector \mathbf{w} can be changed to a unit vector \mathbf{v} in the same direction by dividing \mathbf{w} by its norm:

$$\mathbf{v} = \frac{\mathbf{w}}{\|\mathbf{w}\|}.$$

- Today's topic is how to construct an orthogonal or orthonormal basis.

6.4 The Gram-Schmidt Process

- According to the wikipedia, the Gram-Schmidt Process is named after Jorgen Pedersen Gram (1850-1916) and Erhard Schmidt (1876-1959), but was already known to Pierre-Simon Laplace (1749-1827).
- Given a linear space W , how do we construct an orthogonal or orthonormal basis?
- The Gram-Schmidt Process is one way to answer that question.
- It's based on the assumption that we already have some basis of W and uses that basis to construct an orthogonal basis.
- We start with Example 2 from the textbook (page 356).
- Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

and

$$W = \text{span} \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \}.$$

Construct an orthogonal basis

$$\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$$

of W .

- Actually we are going to do a little more: we are going to get a set of orthogonal vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that

$$\begin{aligned}\text{span}\{\mathbf{x}_1\} &= \text{span}\{\mathbf{v}_1\}, \\ \text{span}\{\mathbf{x}_1, \mathbf{x}_2\} &= \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}, \\ \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} &= \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.\end{aligned}\tag{5}$$

- The order of the basis vectors matters. We have a sequence of nested vector spaces, and nested bases of those spaces.
- Recall that the requirement that the \mathbf{v}_i be orthogonal means that

$$\mathbf{v}_1 \bullet \mathbf{v}_2 = \mathbf{v}_1 \bullet \mathbf{v}_3 = \mathbf{v}_2 \bullet \mathbf{v}_3 = 0. \tag{6}$$

- Clearly, the requirements (5) and (6) do not determine the vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 uniquely. We can multiply them with non-zero constants and the properties (5) and (6) will still be satisfied.
- A common requirement is that the \mathbf{v}_i be **unit vectors**, i.e., $\|\mathbf{v}_i\| = 1$.
- The textbook also suggest the possibility of making the entries of the \mathbf{v}_i integer to simplify hand calculations. For this example, let's follow that suggestion.

- The general process is described by **Theorem 11**, page 357, textbook: Given a basis

$$\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p\}$$

for a non-zero subspace W of \mathbb{R}^n , define

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1} \mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2} \mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \frac{\mathbf{x}_p \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_p \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2} \mathbf{v}_2 - \dots - \frac{\mathbf{x}_p \bullet \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \bullet \mathbf{v}_{p-1}} \mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W . In addition,

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \quad \text{for } k = 1, 2, \dots, p.$$

- This is version 1 of the Gram-Schmidt Process.

- Version 2 is just a more compact notation for the process. For $k = 1, \dots, p$ define

$$\mathbf{v}_k = \mathbf{x}_k - \sum_{i=1}^{k-1} \frac{\mathbf{x}_k \bullet \mathbf{v}_i}{\mathbf{v}_i \bullet \mathbf{v}_i} \mathbf{v}_i. \quad (7)$$

- This works even for $k = 1$ if you use the convention that the empty sum is zero.
- In most references you will find a description of the Gram-Schmidt process that defines **orthonormal vectors** $\mathbf{v}_1, \mathbf{v}_2, \dots$. This means that the \mathbf{v}_i are orthogonal, and that they are also unit vectors. In the usual description of the Gram Schmidt process the \mathbf{v}_i are normalized as soon as they are computed. Thus the denominators $\mathbf{v}_i \bullet \mathbf{v}_i$ equal 1, and disappear.
- This gives rise to Version 3 of the Gram-Schmidt Process:

For $k = 1, \dots, p$ define

$$\begin{cases} \mathbf{w}_k &= \mathbf{x}_k - \sum_{i=1}^{k-1} (\mathbf{x}_k \bullet \mathbf{v}_i) \mathbf{v}_i \\ \mathbf{v}_k &= \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|} \end{cases} \quad (8)$$

The QR Factorization

- Orthonormal bases are particularly important if they form the columns of a matrix.
- Definition: A square matrix Q is **orthogonal** if its columns form an **orthonormal** set.
- This means that

$$Q^T Q = I,$$

i.e., Q is invertible, and

$$Q^{-1} = Q^T.$$

(see textbook, page 346.)



An orthogonal matrix should really be called **orthonormal**, but it's not. There is no such thing as an orthonormal matrix.

- Examples of orthogonal matrices include:
 - The identity matrix
 - A rotation matrix:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- A permutation matrix.
- A **Householder Reflection**, i.e., a matrix of the form

$$H = I - 2\mathbf{u}\mathbf{u}^T$$

where \mathbf{u} is a unit vector.

- It is of course easy to come up with a non-square matrix whose columns form an orthonormal set. For example, pick a subset of the columns of an orthogonal matrix. The columns are still orthonormal.



However, there is no generally accepted name for that kind of matrix. So every time we use a matrix like this we have to describe it as in

- **Theorem 12**, page 359, textbook. If A is an $m \times n$ matrix with linearly independent columns, then A can be factored a

$$A = QR$$

where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{Col}(A)$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

- Let's first revisit Example 1.

- The ideas apply in general. The matrix Q can be constructed in several different ways, including the Gram-Schmidt process. Once we have Q we can construct R by the observation that

$$A = QR \quad \Longleftrightarrow \quad R = Q^T A.$$



But read the numerical notes on page 360 of the textbook.