Math 1320-9    Notes of 1/17/2020

• Of course, no class Monday, ... MLK Day

• Reserved our classroom for Tuesdays, 9:40-10:30. Will stay and answer questions.

• Today last day to drop the class without being charged tuition — I hope you won’t.

8.4 More Convergence Tests

Alternating Series

• An alternating series has terms that are alternately positive and negative.

\[ \sum_{n=1}^{\infty} (-1)^{n-1} b_i = b_1 - b_2 + b_3 - b_4 + b_5 - \ldots \]

where all \( b_n \) are positive (or all \( b_n \) are negative).

• Of particular importance, and simplicity, are alternating series whose terms have decreasing absolute values.

• The alternating harmonic series is

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}. \]
Alternating Series Test

If the alternating series

\[ s = \sum_{n=1}^{\infty} (-1)^{n+1} b_i = b_1 - b_2 + b_3 - b_4 + b_5 - \ldots \quad (b_n > 0) \]

satisfies \( b_{n+1} \leq b_n \) for all \( n \) and \( \lim_{n \to \infty} b_n = 0 \) then the series is convergent. Moreover, denoting the partial sums by

\[ s_n = \sum_{i=1}^{n} (-1)^{i+1} b_i \]

we have

\[ |s - s_n| < b_{n+1} \quad \text{for all } n. \]
• Example. Alternating harmonic series

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}. \]

• Some partial sums

\[ S_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}. \]

\begin{tabular}{cccccccc}
  \textit{n} & 10 & 100 & 1000 & 10,000 & 100,000 & 1,000,000 \\
  \textit{S}_n & 0.6456349 & 0.688172 & 0.692647 & 0.693097 & 0.693142 & 0.6931467 \\
\end{tabular}

Actually

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2 \approx 0.6931471 \]

We will understand this next week ...
• Example 2

\[
\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n - 1}
\]

\[
\frac{d}{dx} \frac{3x}{4x - 1} = \frac{3(4x - 1) - 4(3x)}{(4x - 1)^2} = \frac{-3}{(4x - 1)^2} < 0
\]
• Example 3

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1} \]

\[ \frac{d}{dx} \frac{n^2}{n^3 + 1} = \frac{2n(n^3 + 1) - 3n^2 n^2}{(n^3 + 1)^3} = \frac{-n^4 + 2n}{(n^3 + 1)^2} < 0 \]
Absolute Convergence

- A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.

  ❝ Absolute convergence implies convergence. ❝

- Many of our facts are intuitive, but this one is not (at least not to me).

- So here is the argument from the textbook. It’s a little tricky:
  note that
  
  $$0 \leq a_n + |a_n| \leq 2|a_n|$$

  because $a_n$ is either 0 (if $a_n < 0$) or $2a_n$ (if $a_n > 0$.)

  Thus by the comparison test, $\sum (a_n + |a_n|)$ converges.

  Then

  $$\sum a_n = \sum (a + |a_n|) - \sum |a_n|$$

  is the difference of two convergent series, and therefore convergent.
• The alternating harmonic series is convergent but not absolutely convergent.
• modified Example 7

\[ \sum_{n=1}^{\infty} \frac{\cos n!}{n^2}. \quad \text{converges} \]

• The numerator looks like a random number between \(-1\) and 1 ...

\[ 0 < \left| \frac{\cos n!}{n^2} \right| < \frac{1}{n^2} \]
The Ratio Test

The ratio test effectively compares a series with itself and states, very plausibly, that if a series behaves like a geometric series in the limit then it converges or diverges like a geometric series.

If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \) then the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent (and hence convergent).

If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \) then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L = 1 \) then the series the test is inconclusive.

- Example 8

\[
\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n} = \mathcal{Z}
\]

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{3 \cdot (n+1)^3}{3^{n+1} \cdot n^3}
\]

\[
= \frac{1}{3} \cdot \frac{(n+1)^3}{n^3} \quad \Rightarrow \quad \frac{1}{3} < 1
\]

\[
\frac{(n+1)^3}{n^3} = \frac{n^3 + 3n^2 + 3n + 1}{n^3} = 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3} \quad \Rightarrow \quad \int
\]
• Ratio test applied to series with terms that are rational functions of \( n \).

• Apply to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{1}{n} \).

\[
\frac{\frac{1}{n}}{(n+1)^2} = \frac{n^2}{(n+1)^2} \quad \Rightarrow \quad 1
\]

\[
\frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{n}{n+1} \quad \Rightarrow \quad 1
\]