Announcements

- PA Office hours: after class, in class or JWB 127
- SK Office hours: Tu, 1:30-2:30pm, WEB 1705, and We, 9:30-10:30am, LCB loft

Undergraduate Colloquium

Wednesdays, 12:55-1:45pm, LCB 225
This is a weekly event, and everybody is welcome.

This week’s UGC is relevant to what we are doing in that it deals with infinity. The speaker is different every week, but this week I will speak on Hotel Infinity.

Abstract: You are the owner of Hotel Infinity. It has infinitely many rooms, and it’s full. A new guest arrives and insists you give her a room. How do you accommodate her? The next day, a family with infinitely many members arrives, each of whom wants a private room. The Hotel is still full. The next day infinitely many families, each with infinitely many members, arrive. Each member of each family insists on a private room. What do you do? We’ll use this puzzle as an introduction to some subtleties of the concept of infinity.

Of course you know that hw 1 is due tonight.
Achilles and the Tortoise

- Zeno of Elea ($\approx 495$–$430$BC) stated a number of paradoxes involving infinity that have stimulated mathematical and philosophical thinking for the past two and a half thousand years. One is about Achilles and the tortoise.

- Achilles, of course, was the greatest Greek warrior in the Trojan War. Suppose he enters a race with a tortoise. To be fair, the tortoise gets a head start. Zeno argued that Achilles will never catch up with the tortoise, as follows: Achilles will take some time to reach the point where the tortoise stated. By the time he does the tortoise will have moved ahead, and it will take Achilles some time to reach the point where it is now. When he does the tortoise will have moved ahead. The process repeats indefinitely and Achilles will never pass the tortoise.

- Here is another of Zeno’s paradoxes. Motion is impossible! Suppose you want to move from here to there. Well, first you have to cover half the distance. To get there you have to cover half that distance. And so on. Getting there takes infinitely many steps and you can’t do it!
Figure 1. Zeno of Elea.

Cartoon taken from

https://chadebrack.com/zenos-paradoxes-of-motion-is-motion-possible/
8.2 Series

- A **series** is a sum of infinitely many terms.
- Textbook talks about *infinite* sequences or series. We usually drop the word “infinite”.
- A “finite series” would be simply a sum.
- Examples:
  
  - $\pi = 3.14159 \ldots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \ldots$

  - $0.9999 \ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots = 1.$

  - $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{3}{2}$

  - the last two are examples of a **geometric series**.

  \[ 1 + r + r^2 + r^3 + r^4 + \ldots \]
• Basic idea: start with the sequence of terms. Form the sequence of **partial sums**, i.e., the sums of the first so many terms, define the sum of the series to be the limit of the partial sums.

• More precisely:

• **Definition.** (p. 566) Given a series

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \]

let \( s_n \) denote the \( n \)-th **partial sum**

\[ s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n. \]

If the sequence \( \{s_n\} \) is convergent, i.e.,

\[ s = \lim_{n \to \infty} s_n \]

exists, then the series \( \sum_{n=1}^{\infty} a_n \) is called **convergent** and we write

\[ a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n = s. \]

We call \( s \) the **sum** (or sometimes the **value** or **limit**) of the series. If the sequence \( \{s_n\} \) is divergent we call the series \( \sum_{n=1}^{\infty} a_n \) **divergent**.

• As always, we introduce a new concept by reducing it to an old one. In this case the **sum of the series** is the limit of the **sequence of partial sums**.
Geometric Series

\[ S_n = \frac{1 - r^{n+1}}{1 - r} \]

\[ n = 1 \]
\[ 1 + r = \frac{1 - r^2}{1 - r} \]

\[ (1 + r)(1 - r) = 1 - r^2 = 1 - r + r - r^2 \]

\[ (1 + r + r^2 + \ldots + r^n)(1 - r) = (1 - r^{n+1}) \]

\[ \sum_{i=0}^{\infty} r^i = 1 + r + r^2 + \ldots = \begin{cases} \frac{1}{1-r} & |r| < 1 \\ \text{DNE} & |r| \geq 1 \end{cases} \]

\[ r = 1 \quad S_n = n + 1 \]
\[ r = -1 \]
\[ S_n = 1, 0, 1, 0, 1, 0, \ldots \quad \text{divergent} \]
\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2 \]

\[ r = \frac{1}{2} \]

\[ r = \frac{1}{4} \]

\[ 1 + \frac{1}{4} + \frac{1}{16} + \ldots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \]

\[ r = \frac{1}{2} \]

\[ 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots \]

\[ = 3 \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots \right) = 3 \cdot 2 = 6 \]

\[ r = \frac{1}{2} \]

\[ 1 + \frac{1}{2} + \frac{1}{4} + \ldots = \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots \right) = \frac{1}{2} \cdot 2 = 1 \]

\[ \frac{1}{2} + \frac{1}{4} + \ldots = \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots \right) - 1 = 2 - 1 = 1 \]
First example of a Power Series

\[ \sum_{n=0}^{\infty} x^n = \begin{cases} 
\frac{1}{1-x} & |x| < 1 \\
\text{DNE} & |x| \geq 1 \\
\text{divergent} & 
\end{cases} \]
The Harmonic Series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \]

\[ \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \geq \frac{1}{2} \]

next 8, 16, 32, ... > \frac{1}{2}

\[ \infty \text{ many } > \frac{1}{2} \implies \text{divergence} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

\[ S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \int_{1}^{n+1} \frac{1}{x} \, dx = \ln(n+1) \]
**Theorem 6.** If the series $\sum_{n=1}^{\infty} a_n$ is convergent then

$$\lim_{n \to \infty} a_n = 0$$
Telescoping Sums

\[ S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \]

\[ \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots \]

\[ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1 - n}{n(n+1)} = \frac{1}{n(n+1)} \]

\[ S = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \]
Finitely Many Exceptions Don’t Matter

\[ 2 + 3 + 5 + 7 + 11 + \ldots + 1001 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]

\[ \text{WTI} + 2 \]
Example 9: Compute

\[ S = \sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right). \quad = 4 \]
Convergent or not:

• $\sum_{n=1}^{\infty} \left(\frac{1}{2+n}\right)^n$ converges

\[ 0 < \frac{1}{2+n} < \frac{1}{2} \]
\[ \frac{1}{2^n} < \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \quad 1 < 1 \]

• $\sum_{n=1}^{\infty} \frac{1}{n - \sin^2 n}$ diverges

\[ \frac{1}{n - \sin^2 n} > \frac{1}{n} \]

• $\sum_{n=0}^{\infty} \frac{1}{n!} = e$ converges

\[ \frac{1}{n!} < \frac{1}{2^n} \]
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\[ y = 5x - 3x^2 \]

\[ 5x - 3x^2 = mx \quad 1 \div x \]

\[ 5 - 3x = m \]

\[ 3x = 5 - m \]

\[ x = \frac{5 - m}{3} \]

\[ A = B \]

\[ A + B = \int_0^{5/3} (5x - 3x^2) \, dx \]

\[ = \frac{5}{2} \left[ \frac{5^2}{2} \right]_0^{5/3} - \frac{5^3}{3^3} \]

\[ = \frac{5}{2} \cdot \frac{5^2}{2} - \frac{5^3}{3^3} \]
\[ \frac{5 - m}{3} \]

\[
\int_{0}^{5} 5x - 3x^2 - mx \, dx = \frac{1}{2} \left( \frac{5}{2} \cdot \frac{5^2}{3} - \frac{5^3}{3^3} \right)
\]

**solve for** \( m \)

---

\[ y = ax - bx^2 = 0 \]
\[ x = 0 \quad a, b > 0 \]
\[ x = \frac{a}{b} \]

\[
\int_{0}^{a/b} ax - bx^2 \, dx = A + B
\]

\[
= \left[ \frac{ax^2}{2} - \frac{bx^3}{3} \right]_{0}^{a/b}
\]

\[
= \frac{a^3}{2b^2} - \frac{b a^3}{3 b^3}
\]

\[
= \left( \frac{1}{2} - \frac{1}{3} \right) \frac{a^3}{b^2} = \frac{a^3}{6b^2}
\]

\[ ax - bx^2 = mx \]

\[ a - bx = m \]
\[
\begin{align*}
\frac{b x}{a - m} &= x \\
\frac{a - m}{b} &= x \\
\int_0^1 a x - b x^2 - m x \, dx &= \frac{a}{12 b^2} \\
\frac{a x^2}{2} - \frac{b x^3}{3} - \frac{m x^2}{2} \left[ \frac{x^{a-m}}{a-m} \right]_0^1 &= \frac{a^3}{12 b^2} \\
\frac{a (a-m)^2}{2 b^2} - \frac{b (a-m)^3}{3 b^3} - \frac{m (a-m)^2}{2 b^2} &= \frac{a^3}{12 b^2} \\
\frac{(a-m)^3}{2 b^2} - \frac{(a-m)^3}{3 b^2} &= \frac{a^3}{12 b^2} \\
\frac{1}{6} (a-m)^3 &= \frac{a^3}{12} \\
(a - m)^3 &= \frac{a^3}{2} \\
a - m &= \frac{a}{2^{\frac{3}{2}}} \\
m &= a - \frac{a}{2^{\frac{3}{2}}} = \left(1 - \frac{1}{2^{\frac{3}{2}}} \right) a
\end{align*}
\]
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\[ p = c v^2 + \frac{d}{v^2} \quad c, d > 0 \]

\[ \min \text{ pwr.} \quad \dot{p} = 2cv - \frac{2d}{v^3} = 0 \quad \frac{1}{2} \cdot \frac{v^3}{2} \]

\[ cv^n = d \]

\[ v^n = \frac{d}{c} \]

\[ v = \left( \frac{d}{c} \right)^{1/n} \]

\[ \text{distance} = k v \cdot \frac{1}{p} \quad k \text{ constant of proportionality} \]

\[ \max \quad \frac{v}{p} = \frac{v}{c v^2 + \frac{d}{v^2}} = \frac{v^3}{c v^4 + d} \]
\[
\left( \frac{v}{p} \right)' = \frac{3v^2 (cv^4 + d) - v^3 (4cv^3)}{(cv^5 + d)^2} = 0
\]

\[3(cv^4 + d) - v^4 cv^3 = 0\]

\[cv^4 = 3d\]

\[v^4 = \frac{3d}{c}\]

\[v = \left( \frac{d}{c} \right)^{1/4}\]
\[
\frac{s}{h} = \frac{d+s}{H} \quad \text{d, H constants}
\]

\[
\frac{s' h - s h'}{h^2} \leq \frac{s'}{H}
\]

\[
\frac{s'}{h} - \frac{s h'}{h^2} = \frac{s'}{H}
\]

\[
s' \left( \frac{1}{h} - \frac{1}{H} \right) \leq \frac{s h'}{h^2}
\]

\[
s' \leq \frac{s h'}{h^2} \left( \frac{1}{h} - \frac{1}{H} \right) = \frac{s h' - h H}{h^2 (H - h)}
\]

\[
= \frac{s h' H}{h (H - h)}
\]

\[
= \frac{(d+s) h' H}{H (H - h)} \leq \frac{d+s}{H} \cdot h'
\]

\[
\frac{d+s}{H} \cdot v = \frac{dv}{H}
\]