12.7 Triple Integrals

- The ideas we discussed for double integrals carry over quite naturally to triple integrals, i.e., integrals over a subset of three dimensional space.

- The physical interpretation is perhaps not so straightforward.

- Single integrals give area.
- Double integrals give volume.
- Triple integrals?

- Four dimensional volume is unintuitive to most of us. However, for example, integrating density gives mass.

- We could have a Riemann sum definition of triple integrals but it would offer little that’s new. So let’s jump right in.

- Example. Suppose

  \[ f(x, y, z) = x^2yz \]

  and we want to integrate over the box (technically a rectangular cuboid)

  \[ B = \{(x, y, z) : 1 \leq x \leq 2, \ 0 \leq y \leq 1, \ 0 \leq z \leq 2\} \]

- The triple integral of \( f \) over \( B \) is denoted by

  \[
  I = \iiint_B x^2yz \, dv = \int_0^2 \int_0^1 \int_1^2 x^2yz \, dx \, dy \, dz.
  \]

- There are 6 possible orders of integration:

  \[ xyz, \ xzy, \ yxz, \ yzx, \ zxy, \ zyx. \]
• All give the same answer, i.e.,

\[ I = \frac{7}{3} \]

• Let’s do one of them.

\[
I = \iiint_1^2 x^2yz \, dx \, dy \, dz
\]

\[
= \iiint_0^1 \left[ \frac{x^3}{3} \right]^2 \, yz \, dy \, dz
\]

\[
= \iiint_0^1 \frac{7}{3} \, yz \, dy \, dz
\]

\[
= \frac{7}{3} \int_0^1 \left[ \frac{yz^2}{2} \right]_0^1 \, dz
\]

\[
= \frac{7}{3} \int_0^1 z \, dz
\]

\[
= \frac{7}{6} \left[ \frac{z^2}{2} \right]_0^2 = \frac{7}{3}
\]
• More general regions can be very tricky.

• Example:

\[ I = \int_{-2}^{5} \int_{0}^{3x} \int_{y}^{x+2} f(x, y, z) \ldots \, dz \, dy \, dx \]

• In what sequence do we need to integrate for this to make sense? In other words, how do we fill in the three dots?

• The limits of integration may be tricky to determine. It may be possible, or perhaps even necessary, to change the sequence of integration, but then the limits of integration will change in subtle ways. It would be fun to explore these issues, but they are beyond the scope of our class.

**Center of Mass**

• Suppose \( S \) is some three dimensional solid and \( \delta(x, y, z) \) is the density of that solid at the point \( (x, y, z) \). Then the center of gravity of \( S \) is the point \( (\bar{x}, \bar{y}, \bar{z}) \) where

\[
\begin{align*}
    m &= \iiint_S \delta(x, y, z) \, dV \\
    \bar{x} &= \frac{1}{m} \iiint_S x \delta(x, y, z) \, dV = \frac{M_{yz}}{m} \\
    \bar{y} &= \frac{1}{m} \iiint_S y \delta(x, y, z) \, dV = \frac{M_{xz}}{m} \\
    \bar{z} &= \frac{1}{m} \iiint_S z \delta(x, y, z) \, dV = \frac{M_{xy}}{m}
\end{align*}
\]

More than three integrals

• There is of course nothing to stop us from have more than 3 integrals.
• An application is in probability theory. Suppose $S$ is a region in $n$-dimensional space and $z = f(x_1, \ldots, x_n)$ is a given probability density functions satisfying

$$f(x_1, \ldots, x_n) \geq 0$$

and

$$\int\int\ldots\int_R f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n = 1.$$

• Then the expected value of a function $g$ on $S$ is

$$E(g) = \int\int\ldots\int_R f(x_1, \ldots, x_n) g(x_1, \ldots, x_n) \, dx_1 \ldots dx_n.$$

### 12.8 Triple Integrals in Cylindrical and Spherical Coordinates

• We start with cylindrical coordinates. Recall that cylindrical just means polar in $x$-$y$, leaving $z$ as before:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2$$

and

$$f(x, y, z) = f(r \cos \theta, r \sin \theta, z).$$

• The basic idea is simply to add an integration in $z$ to the integration in polar coordinates. We get, for example:\

$$\int\int\int_S f(x, y, z) \, dv = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r, \theta)}^{g_2(r, \theta)} rf(r \cos \theta, r \sin \theta, z) \, dz \, dr \, d\theta.$$\n
\[\text{Note the extra factor } r.\]

\[\text{Why do I say “for example”?}\]
• Example: Find the mass and center of mass of a right circular cylinder of height $h$ and radius $a$, assuming that the density is proportional to the distance from the base.

$$\delta(x, y, z) = k \cdot z \quad k = 1$$

$$\bar{x} = \bar{y} = 0$$

$$\frac{h}{2} < \bar{z} < h$$

$$m = \iiint_{D} z \cdot r \, dz \, dr \, d\theta$$

$$= 2\pi a \int_{0}^{h} \int_{0}^{a} \frac{z^2}{2} \, dr \, dz$$

$$= 2\pi a \int_{0}^{h} \left[ \frac{z^2r}{2} \right]_{0}^{a} \, dz$$

$$= 2\pi a \int_{0}^{h} \frac{a^2}{2} \, dz$$

$$= \pi a^2 \frac{h^2}{2}$$

$$x = \frac{\pi a^2 h^2}{6}$$

$$\bar{z} = \frac{\pi a^2 h^2}{6} \frac{\frac{2}{3}}{\frac{2}{3}} = \frac{h}{3} > \frac{h}{2}$$

• Expectations?

• The important thing to realize is that the volume element $dx\,dy\,dz$ is replace with $rdrd\theta\,dz$.

• That’s just like polar coordinates. Spherical coordinates are more complicated.
Spherical Coordinates

- See Figure 1 (which is Figure 6 on page 885 of the textbook) for the definition of spherical coordinates.
  - \( \theta \) is the angle with the \( x \)-axis in the \( x-y \) plane
  - \( \phi \) is the angle with the \( z \)-axis
  - \( \rho = \sqrt{x^2 + y^2 + z^2} \) is the distance from the origin.
  - We can go from spherical to Cartesian via
    
    \[
    x = \rho \sin \phi \cos \theta, \\
    y = \rho \sin \phi \sin \theta, \\
    z = \rho \cos \phi
    \]
• Now consider a small region corresponding to

\[ \Delta \rho, \quad \Delta \theta, \quad \Delta \phi. \]

Figure 2. Volume Element.

• As illustrated in Figure 2 (which is Figure 7 on page 885 of the textbook) its volume is approximately

\[ \Delta V \approx \rho^2 \sin \phi \Delta \rho \Delta \theta \Delta \phi. \]
• Thus we get the lovely formula

\[ \iiint_{S} f(x, y, z) dV = \]
\[ = \iiint_{S} f(\rho \sin \phi \cos \theta, \rho \sin \phi \cos \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi \]

with appropriated limits of integration on the individual integrals on the right hand side of that equation.

• Let’s finish our discussion of multiple integrals with the following somewhat involved problem which I took from another Calculus book. For a solid sphere of radius \( a \) [assume \( a = 1 \)], find each average distance [of its points]
(a) from its center,
(b) from a diameter,
(c) from a point on its boundary.

• This is a bit cryptic. Plus, for item (c) there is the perhaps even more cryptic hint: Consider

\[ \rho = 2a \cos \phi. \]

• Way back in 1310 we defined the **average value** of a function \( f \) on and interval \([a, b]\) to be

\[ \text{avg}(f) = \frac{\int_{a}^{b} f(x) dx}{b - a} \]

• The natural generalization of this to triple integrals is

\[ \text{avg}(f) = \frac{\iiint_{S} f(x, y, z) dA}{\iiint_{S} dA} \]

• As an exercise, think of this formula in terms of a collection of finitely many equally distributed small particles of the same mass (like water molecules, say).
• Instead of \( f \) we have the distance from the given object.

• For the distance from the center we expect that distance to be greater than \( 1/2 \) since the area of the region of a spherical shell increases as its radius increases.

• Let \( S \) denote the unit sphere, and note that \( \rho \) is the distance of a point from the origin. We get:

\[
\text{avg}_{\text{center}} = \frac{\iiint_S \rho \, dV}{\iiint_S \, dV} = \frac{\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi \rho \, d\rho \, d\phi \, d\theta}{4\pi/3} = \frac{3}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^3 \sin \phi \rho \, d\rho \, d\phi \, d\theta = \frac{3}{4\pi} \times \frac{1}{4} \times \frac{1}{4} \times 2\pi \times 2 = \frac{3}{4},
\]

which is plausible.

• To compute the average distance from a diameter let’s suppose the diameter is contained in the \( z \)-axis. The distance of a point \((x, y, z)\) from the \( z \)-axis is of course

\[
r = \sqrt{x^2 + y^2} = \rho \sin(\phi).
\]

The average distance is harder to guess because as we move away from the \( z \)-axis the cylindrical shells consisting of points of equal distance have and increasing radius balanced by a decreasing height. We expect and average distance in the vicinity of \( 1/2 \). Proceeding as before we get
\[
\text{avg diameter} = \frac{\iiint_S \rho dV}{\iiint_S dV} \\
= \frac{3}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^3 \sin^2 \phi \rho d\rho d\phi d\theta \\
= \frac{3}{4\pi} \times \int_0^1 \rho^3 d\rho \times \int_0^\pi \sin^2 \phi d\phi \times \int_0^{2\pi} d\theta \\
= \frac{3}{4\pi} \times \frac{1}{4} \times \frac{\pi}{2} \times 2\pi \\
= \frac{3\pi}{16} \\
\approx 0.589
\]

which again is plausible.

• Finally let’s compute the average distance of all points from specific point on the boundary. Think of the average distance of all points on and inside the earth from Salt Lake City. Points are up to 2 radii, i.e., 2 on the unit sphere away, so we expect the average to be around 1.

• What about the hint: consider

\[ \rho = 2 \cos \phi? \]

That’s actually the equation of the sphere of radius 1 around the point (0,0,1). To see this multiply with \( \rho \) on both sides to get

\[ \rho^2 = 2\rho \cos \phi \]

which, in Cartesian coordinates, becomes

\[
x^2 + y^2 + z^2 - 2z = 0
\]

or

\[
x^2 + y^2 + (z - 1)^2 = 1.
\]

• Pretty cool.
Notice that $\phi$ ranges only from 0 to $\pi/2$ because $\rho$ cannot equal a negative value.

- The distance of a point $(x, y, z)$ on that sphere from the origin is
  $$\sqrt{x^2 + y^2 + z^2} = \rho.$$  

- So we get
  \[
  \text{avg}_{\text{bdry}} = \frac{3}{4\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos \phi} \rho^3 \sin \phi \rho \, d\rho \, d\phi \, d\theta \\
  = \frac{3}{4\pi} \int_0^{2\pi} \int_0^{\pi/2} \frac{16}{4} \cos^4 \phi \sin \phi \, d\phi \, d\theta \\
  = \frac{3}{\pi} \int_0^{2\pi} \left[ -\frac{1}{5} \cos^5 \phi \right]_0^{\pi/2} \, d\theta \\
  = \frac{3}{\pi} \int_0^{2\pi} \frac{1}{5} \, d\theta \\
  = \frac{6}{5}.
  \]