12.5 Probability

- Probability is a big subject. You can take year long courses (like our Math 5010-20) on it!
- We’ll just scratch the surface.
- If you roll a die you have 6 possible outcomes, rolling a 1, 2, 3, 4, 5, or 6, that all have the same probability, namely $\frac{1}{6}$. The sum of the probabilities is 1.
- The probability that you roll a 2 or a 3 is the sum of the probabilities of rolling a 2 or a 3, i.e.,
  \[ \frac{1}{6} + \frac{1}{6} = \frac{1}{3}. \]
- This concept of probability can be transferred to the continuous case.

One of the great simplifying principles of Analysis is that integrals behave much like sums.
- We’ll just list the definitions and work a couple of examples.
- Suppose you have a die that returns a random number $X$ in the interval $[0, 6]$. Then the probability that the random $X$ is in the interval $[1.2, 3.3]$, for example, is
  \[ P(1.2 \leq X \leq 2.3) = \frac{1}{6} \times (3.3 - 1.2) = \frac{2.1}{6} = \int_{1.2}^{3.3} \frac{1}{6} \, dx. \]
A probability density function $f$ of a continuous random variable $X$ is a function such that:

a. $f(x) \geq 0$ for all real numbers $x$,

b. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Then the probability that $X$ is in the interval $[a, b]$ is the

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx.$$ 

These concepts are introduced in section 6.8.

In section 12.5 they are generalized to functions of two variables.

Suppose we have two random variables $X$ and $Y$, such as the lifetimes of two components of a machine, or the height and weight of a piece of equipment.

Then the joint density function of $X$ and $Y$ is a function $f$ of two variables such that

a. $f(x, y) \geq 0$ for all $x$ and $y$

b. $\iint_{\mathbb{R}^2} f(x, y)\,dA = 1$.

If $D$ is a region in $\mathbb{R}^2$ then the probability that $(X, Y)$ is in $D$ is simply

$$P((X, Y) \in D) = \iint_{D} f(x, y)\,dA.$$ 

In the particular case that $D$ is a rectangle we get

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_{a}^{b} \int_{c}^{d} f(x, y)\,dy\,dx.$$
Example 5, p. 864, textbook: Suppose the joint density function for $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 \leq 10, 0 \leq y \leq 10, \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $C$ such that $f$ is a joint density function, and compute the probability that $X \leq 7$ and $Y \geq 2$. 
• If $f$ is a probability density function then its **expected value** is

$$
\mu = \int_{\mathbb{R}} x f(x) \, dx.
$$

**Figure 1.** Exponential PDFs with expected values 1, 2, 3,..

• Example: Show that for positive $\mu$ the function

$$
f(t) = \begin{cases} 
\mu^{-1} e^{-t/\mu} & t \geq 0 \\
0 & \text{otherwise}
\end{cases} \quad (1)
$$

is a probability density function and compute its expected value.

• Figure 1 shows exponential probability density functions with expected values of 1, 2, 3.
• If we have a joint density function we can apply the concept of expected value to each variable independently. We get

\[ \mu_1 = \iint_{\mathbb{R}^2} xf(x,y)dA \quad \text{and} \quad \mu_2 = \iint_{\mathbb{R}^2} yf(x,y)dA. \]

Note the similarity with centers of mass! This is of course no coincidence.

• Example 6, page 864, textbook. The manager of a movie theater determines that the average time moviegoers wait in line to buy a ticket for this week’s film is 10 minutes and the average time the wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less that 20 minutes before taking his or her seat. Assume both waiting times have probability density functions of the form (1).