11.8 Lagrange Multipliers

- We start with an example: Suppose we want to maximize

\[ f(x, y) = 2x + 3y \]

over the set of all points \( (x, y) \) for which

\[ g(x, y) = x^2 + y^2 \leq 1. \]

Figure 1. Unit circle and contour lines of \( f(x, y) = 2x + 3y \).

- Figure 1 shows the contour lines of \( f \) and the unit circle.
Figure 2. Copy of Figure 1.

- Clearly at the point where we have the maximum the gradient of $f$ must be orthogonal to the contour line of $g$.

That means that the gradient of $f$ must be a scalar multiple of the gradient of $g$!

- $\nabla f = \lambda \nabla g$

- $\lambda$ is a Lagrange Multiplier.
• Work problem formally: Minimize

\[ f(x, y) = 2x + 3y \quad \text{subject to} \quad g(x, y) = x^2 + y^2 \leq 1. \]

\[ \nabla f = \langle 2, 3 \rangle = 2\langle 2x, 2y \rangle \]

\[ \begin{align*}
2 &= 22x \\
3 &= 22y \\
\frac{2}{3} &= \frac{22y}{22x} = \frac{y}{x}
\end{align*} \]

\[ x + y^2 = 1 \]

\[ \begin{align*}
2 + \frac{9}{4}x^2 &= 1 \\
x^2 &= \left(1 + \frac{9}{4}\right) = 1 \\
\frac{13}{4} &= x^2 \\
x &= \frac{4}{13} \quad \Rightarrow \quad y = \pm \frac{3}{2} \sqrt{\frac{4}{13}} \\
&= \pm \sqrt{\frac{9}{13}} \\
\end{align*} \]

\[ \left( \sqrt{\frac{4}{13}}, \sqrt{\frac{9}{14}} \right) \quad \text{max} \]

\[ \left( -\sqrt{\frac{4}{13}}, -\sqrt{\frac{9}{14}} \right) \]

\[ f(x, y) = 2 \sqrt{\frac{4}{13}} + 3 \sqrt{\frac{9}{13}} \]
• Example 1: A rectangular box without a lid is to be made from 12m² of cardboard. Find the maximum volume of that box, and the dimensions that give that volume.

• Expectations?

\[ \nabla = \nabla f(x, y, z) = x \ y \ z \\
12 = A = x \ y + 2 \ yz + 2xz = g(x, y, z) \\
\nabla f = \nabla g \\
\n\nabla f = \langle yz, xz, xy \rangle = \nabla (x + 2z, x + 2z, 2x + 2y) \\
xz = \lambda (x + 2z) \quad yz = \lambda (y + 2z) \quad xy = \lambda (2x + 2y) \\
xz = \lambda (x + 2z) \quad yz = \lambda (y + 2z) \quad xy = \lambda (2x + 2y) \\
\begin{align*}
\lambda x + 2xz &= \lambda x + 2yz \\
st &= 1
\end{align*}

\begin{align*}
\lambda x^2 + 2xz &= 2xz + 2yz \\
r &\quad= 2xz \\
2z &= x = y \quad z = \frac{1}{2} x \\
x^2 + 2yz + 2xz &= 12 \\
4z^2 + 4z^2 + 4z^2 &= 12 \\
z^2 &= 1 \quad z = 1 \quad x = y = z
\end{align*}

\[ V = 1 \cdot 2z = 4 \ m^3 \]
Example 4: Find the points on the sphere \( x^2 + y^2 + z^2 = 4 \) that are closest and farthest from the point \((3, 1, -1)\). Do this first without Calculus, and then use Lagrange Multipliers to obtain the same result.

Let \( M = (x, y, z) \) be a point on the sphere.

\[
f(x,y,z) = d^2 = (3-x)^2 + (1-y)^2 + (z+1)^2
\]

\[
g(x,y,z) = x^2 + y^2 + z^2 = 4
\]

\[
\nabla f = \lambda \nabla g
\]

\[
\nabla f = \langle -2(3-x), -2(1-y), 2(z+1) \rangle = \lambda \langle 2x, 2y, 2z \rangle
\]

\[
-3 + x = 2\lambda x \\
-1 + y = 2\lambda y \\
1 + z = 2\lambda z
\]

\[
x - 2x = 3 \\
y - 2y = 1 \\
z - 2z = -1
\]

\[
x = 3/(1-2) \\
y = 1/(1-2) \\
z = -1/(1-2)
\]

\[
x = 6/\sqrt{11} \\
y = 2/\sqrt{11} \\
z = -2/\sqrt{11}
\]

\[
x^2 + y^2 + z^2 = \frac{1}{(1-z)^2} \left( 3^2 + 1^2 + (-1) \right)^2 = 4
\]

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\[
\frac{11}{(1 - 2)^2} = 4 \\
\frac{1}{1 - 2} = \frac{\sqrt{4}}{1 - 2} = \frac{2}{-1} = -2
\]
• What about having 2 (or more) constraints?

\[ \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \]

\[ \lambda_1, \lambda_2 \geq 0 \]

The c.l. of f