• Remainder of this chapter: finding extreme values of a function.

• Remember how we do this for a function of one variable:

\[ f(x) = \min \]

• Extreme (minimum or maximum) values can occur only at three kinds of values of the independent variable: (also called points)

1. **stationary points** where the derivatives is zero:

\[ f'(x) = 0. \]

2. **singular points** where the derivative does not exist.

3. **boundary points** of intervals.

• Practically speaking, there usually are no singular points (the function is everywhere differentiable) and it’s obvious or irrelevant what happens at boundary points.

• So we differentiate, set to zero, solve the equation, and then figure out whether we have a maximum, a minimum, or neither.

There are two major changes for functions of several variables:

1. Several partial derivatives must be zero. (That’s easy to incorporate.)

2. the boundary of a region in 2 or higher dimensional space contains infinitely many points, rather than just two, like an interval. (This is a major issue).
11.7 Local Maximum and Minimum Values

- A function $f$ of two variables has a local maximum at $(a, b)$ if $f(x, y) \leq f(a, b)$ when $f(x, y)$ is near $(a, b)$. The number $f(a, b)$ is called a local maximum value.

- Similarly for minimum, exercise.

- Instead of local can also use relative

- **Global or Absolute Maximum or Minimum Value:** $f(a, b) \geq (\leq) f(x, y)$ for all points $(x, y)$ in the domain of $f$.

- Minimum and maximum values are collectively called extreme values.
• Pretty obvious: If $f(a, b)$ is a maximum value and $f$ is differentiable there then

$$\nabla f(a, b) = 0.$$ 

• The textbook calls this Fermat’s Theorem.

• A point $(a, b)$ is called a critical point if $\nabla f(x, y) = 0$ or if one of the partial derivatives does not exist there.

• Should also include boundary points.

• So to find extrema find critical points and check them out, just like in one variable.
• Example 1: Find the absolute minimum of

\[ f(x, y) = x^2 + y^2 - 2x - 6y + 14 \]

in two different ways.
Example 2: Find extreme values of

\[ f(x, y) = x^2 - y^2. \]
• Recall that in one variable we have a **second derivative test**.

• $f(a)$ is a maximum value of $f'(a) = 0$ and $f''(a) < 0$.

• Similarly for minimum value.

• You’ll learn a very simple second derivative test for functions of several variables in linear algebra (the matrix of second order partial derivatives is positive definite).

• In the mean time, however, here is the test from the textbook:

• Suppose the second partial derivatives are continuous on a disk with center $(a, b)$ and suppose $\nabla f(a, b) = 0$. Let

\[
D = D(a, b) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} (a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b).
\]

Then:
(a) If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
(c) If $D < 0$ then $f(a, b)$ is not an extreme value.
(d) If $D = 0$ the test is inconclusive.
Example 3: Find and classify the critical points of

\[ f(x, y) = x^4 + y^4 - 4xy + 1. \]
Figure 1. Contourplot of Example 3.
• New idea: extreme values on the boundary.

• Example 7: Find the extreme values of

\[ f(x, y) = x^2 - 2xy + 2y \]

on the rectangle

\[ D = \{(x, y) \mid 0 \leq x \leq 3, \; 0 \leq y \leq 2\} \]
• How about finding extrema on the unit circle?