• Recall the idea of linear approximation:

\[ \Delta f = f(x+\Delta x, y+\Delta y) - f(x, y) \approx f_x(x, y)\Delta x + f_y(x, y)\Delta y \]

• The approximation gets the better the smaller \( \Delta x \) and \( \Delta y \).

• The changes of \( f \) due to changes of \( x \) and \( y \) simply add up!

11.5 The Chain Rule

• Recall the chain rule from Math 1310. Let

\[ u = g(x). \]

\[ \frac{df}{dx}(g(x)) = f'(g(x))g'(x) = \frac{df}{du}\frac{du}{dx}. \]

• The derivative of the composition is the product of the derivatives.

• If we have a function of several variables the formulas get more complicated.

• Chain Rule, Version 1. Suppose

\[ x = g(t), \quad y = h(t), \quad \text{and} \quad z = f(x, y) = f(g(t), h(t)). \]

Then

\[ \frac{dz}{dt} = f_x(x, y)\frac{dx}{dt}(t) + f_y(x, y)\frac{dy}{dt}(t) \]

\[ = \frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt} \]

\[ = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \]

\[ = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \]

(1)
• Any of the three right hand sides in (1) can be used equally. Use whatever is most convenient and appropriate for your purposes!

• The equations (1) are eminently plausible: Changing $t$ changes $x$ and $y$. Each of those changes $f$. Each of the two changes of $f$ obeys the ordinary chain rule. The two changes just add, as we saw yesterday in our discussion of linear approximation.

• Note that we use the notation $\partial/\partial$ to denote the partial derivative of a function of several variables, and $d/d$ to denote the derivative of a function of a single variable.

• Example 1:

$$z = x^2y + 3xy^4, \quad x = \sin 2t \quad \text{and} \quad y = \cos t.$$  

Compute $z'$ when $t = 0$.  

• **The Chain Rule, version 2.** Suppose $x$ and $y$ depend on two variables $s$ and $t$. So we have

$$z = f(x, y), \quad x = g(s, t), \quad \text{and} \quad y = h(s, t).$$

Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

• Terminology: In this context, $z$ is the **dependent variable**, $s$ and $t$ are the **independent variables**, and $x$ and $y$ are **intermediate variables**.

• **Example 3:**

$$z = e^x \sin y, \quad x = st^2, \quad \text{and} \quad y = s^2 t.$$  

Find $\partial z/\partial s$ and $\partial z/\partial t$ and express the answers in terms of $s$ and $t$.  

• The Chain Rule, version 3

Suppose that $u$ is a differentiable function of the $n$ variables $x_1, x_2, \ldots, x_n$ and each $x_i$ is a differentiable function of the $m$ variables $t_1, t_2, \ldots, t_m$. Then

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial t_i}, \quad i = 1, 2, \ldots, n$$

Could do examples, but with many independent and intermediate variables this gets tedious.

Implicit Differentiation

• Suppose $y = y(x)$ is a function defined by the equation

$$F(x, y) = 0.$$ 

Find an expression for $y'(x)$ in terms of partial derivatives of $F$. 

• Example 8: Find $y'$ in terms of $x$ and $y$ if

$$x^3 + y^3 = 6xy.$$ 

Compute the value of that derivative when $x = y = 3$. 
• Suppose $z = f(x, y)$ defined by

$$F(x, y, z) = 0$$

Find expressions for $\partial z/\partial x$ and $\partial z/\partial y$. 
Example 9 (modified): Find $z_x$ and $z_y$ if

$$x^3 + y^3 + z^3 + 6xyz = 9.$$ 

Find numerical values for these derivatives when

$$x = y = z = 1.$$
Another Example. Suppose

\[ \phi(t) = f(x_0 + ta, y_0 + tb) \]

What is \( \phi'(t) \) and what is its geometric meaning?