11.2 Limits and Continuity

• We saw on Friday that indeterminate expressions like $0/0$ can be tricky.

• Limits can be defined rigorously, although the textbook does not do so.

• For completeness, without pursuing the matter further, here is a precise definition.

  We say that

  $$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$$

  if for all $\epsilon > 0$ we can find a $\delta > 0$ such that

  $$0 < \| (x-x_0, y-y_0) \| < \delta \implies |f(x,y) - L| < \epsilon$$

• As in the case of one variable, we say that a function $f$ is **continuous at a point** $(x_0, y_0)$ if

  $$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0,y_0)$$

• We say that $f$ is **continuous** or **continuous everywhere** if it is continuous at every point in its domain.

• As in the case of functions of one variable, the usual limit laws hold.
• Also as in the case of functions of one variable, the sum, difference, product, composition, and quotient if the denominator is non-zero, of continuous functions are continuous.

• Polynomials, rational functions (so long as the denominator is non-zero), sin, cos, the exponential, are all continuous.

• tan and logarithms are continuous everywhere in their domain.
11.3 Partial Derivatives.

- We define partial derivatives of functions of two variables, with obvious modifications for functions of more than two variables.

- Suppose \( z = f(x, y) \). We define the partial derivatives of \( f \) with respect to \( x \) and \( y \), respectively, as

\[
fx(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]

and

\[
fy(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.
\]

- Thus we proceed exactly as we discussed: we differentiate with respect to one variable, keeping the other constant.

- Some equivalent notations:

\[
f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} = D_x f = D_1 f.
\]

- The symbol \( \partial \) is pronounced partial as in “partial \( f \) with respect to \( x \)” or “partial \( f \) partial \( x \)”. 

\[
f \equiv \frac{df}{dx} = \frac{d}{dx} f = \partial f = D_x f
\]
• Example 1:

\[ f(x, y) = x^2 + x^2y^3 - 2y^2. \]

\[ f_x = 2x + 2xy^3 \]
\[ f_y = 3x^2y^2 - 4y \]

\[ f_{xx} = 2 + 2y^3 \]
\[ f_{xy} = 6xy^2 \]
\[ f_{yx} = 6xy^2 \]
\[ f_{yy} = 6x^2y - 4 \]
Geometric Interpretation of Partial Derivatives.

- To interpret $f_x(a, b)$ intersect the graph of $z = f(x, y)$ with the plane $y = b$. The graph of $f$ intersects that plane in a curve, and $f(x, y)$ is slope of the tangent of that curve at $x = a$.

- Similarly for $f_y(a, b)$.

- Looking ahead: the two tangent lines define the tangent plane.

- Example:

\[ f(x, y) = 4 - x^2 - 2y^2 \]

Figure 1. Contour lines for $f(x, y) = 4 - x^2 - 2y^2$. 
As in the case of functions of one variable we can compute partial derivatives implicitly.

Example:

\[ x^2 + y^2 + z^2 = 4, \quad z = f(x, y), \quad f_x(1, 1) =? \]

\[ z = \sqrt{4 - x^2 - y^2} = f(x, y) \]

\[ 2x + 2z \frac{\partial z}{\partial x} = 0 \]

\[ \frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z} \]

\[ \frac{\partial z}{\partial x}(1, 1) = -\frac{1}{z} = -\frac{1}{\sqrt{2}} \]

\[ z = f(1, 1) = \sqrt{4 - 1 - 1} = \sqrt{2} \]
More Variables

- Compute all first order partial derivatives of

\[ f(x, y, z) = xe^{yz^2}. \]

\[ \frac{d}{d\ell} e^\ell = e^\ell \]

\[
\begin{align*}
    f_x &= e^{yz^2} \\
    f_y &= xz^2 e^{yz^2} \\
    f_z &= x^2yz e^{yz^2} \\
    &= 2xyze^{yz^2}
\end{align*}
\]
Higher Derivatives

- How many partial derivatives of 
  
  \[ f(x, y, z) = xyz^2. \]

  are non-zero? Compute them.

  \[
  \begin{align*}
  f_x &= yz^2 \\
  f_y &= xz^2 \\
  f_z &= 2xyz \\
  f_{xy} &= z^2 \\
  f_{xz} &= 2yz \\
  f_{yz} &= 2xz \\
  f_{zz} &= 2yx \\
  f_{xyz} &= 2z \\
  f_{xzz} &= 2y \\
  f_{yzz} &= 2x \\
  f_{xxyz} &= 2 \\
  \end{align*}
  \]