10.1 Vector Functions and Space Curves

- A vector function or vector valued function is a function whose domain is a set of real numbers and whose range is a set of vectors.

- We are interested in particular in functions whose ranges are sets in 2-space or 3-space. We write

\[ \mathbf{r}(t) = \langle f(t), g(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} \]  

(1)

in 2-space or

\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \]  

(2)

in 3-space.

- In this context the independent variable \( t \) is also called a parameter. You can think of \( t \) as time and \( \mathbf{r}(t) \) at your location at time \( t \).

- The graph of \( \mathbf{r}(t) \) is the set of all points \( \mathbf{r}(t) \) where \( t \) runs through the domain of \( \mathbf{r} \).

- It’s also, and more frequently, called a plane curve in 2-space or a space curve in 3-space. It’s also sometimes called a trajectory or an orbit. Usually the word “orbit” describes a periodic trajectory.

- As usual, given an equation of the form (1) or (2) we consider the domain of \( \mathbf{r} \) to be the set of all \( t \) for which all components of \( \mathbf{r} \) can be evaluated. This is also called the natural or maximal domain.

- Example 1. What is the natural domain of

\[ \mathbf{r} = \langle t^3, \ln(3 - t), \sqrt{t} \rangle \]

**Domain** = \([0, 3]\)
• Examples (2-space):

\[ \mathbf{r}(t) = \langle \cos t, \sin t \rangle \]

\[ \mathbf{r}(t) = \langle \cos t, \cos t \rangle \]

\[ \mathbf{r}(t) = \langle \cos^2 t, \cos^2 t \rangle \]
\[ r(t) = \langle \cos t^2, \sin t^2 \rangle \]
\[ \mathbf{r}(t) = \langle t \cos t, t \sin t \rangle \]
Example 3 (3-space)

\[ \mathbf{r}(t) = <1 + t, 2 + 5t, -1 + 6t> \]

\[ = <1, 2, -1> + t <5, 6, 6> \]
Example 4 (3-space), see Figure 2, page 695, textbook.

\[ \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \]
Limits and Continuity

• The concepts of limits and continuity are simply applied component by component.

• Suppose

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (3)$$

as before. Then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided all three limits on the right exist.

• The function $\mathbf{r}(t)$ is continuous at a point $a$ if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

• This is equivalent to saying that $\mathbf{r}$ is continuous at $a$ if all three functions $f$, $g$ and $h$ are continuous at $a$. 
Parametrizing a space curve \[ z = 2 - y \]

• Example 6: The plane \( y + z = 2 \) intersects the cylinder \( x^2 + y^2 = 1 \) in an ellipse. Find a parametric representation of the ellipse.

\[
\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle
\]

\[
\langle \cos t, \sin t, 2 - \sin t \rangle
\]

\[
\langle \sin^2 t, \cos^2 t, 2 - \cos^2 t \rangle
\]

• The textbook uses “parametrize” or “parametrization”. Instead you can also use “parameterize” or “parameterization”.

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\[ x^2 + y^2 = 1 \quad y = \sqrt{1 - x^2} \]

\[ y + z = 2 \]

\[ y = 2 - z = \sqrt{1 - x^2} \]

\[ z = 2 - \sqrt{1 - x^2} \]

\[ y = \sqrt{1 - x^2} \]

\[ x = t \]

\[ y = \frac{t}{2}\sqrt{1 - t^2} \]

\[ z = 2 \pm \frac{1}{2}\sqrt{1 - t^2} \]
with(plots);

plotsetup(gif, plotoutput='38plot0.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
plot([cos(t), sin(t), t=0..2*Pi], thickness=3);

plotsetup(gif, plotoutput='38plot1.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
plot([sin(2*t), sin(3*t), t=0..2*Pi], thickness=3);

plotsetup(gif, plotoutput='38plot2.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
plot([sin(2*t), sin(5*t), t=0..2*Pi], thickness=3);

plotsetup(gif, plotoutput='38plot3.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
plot([sin(Pi*t), sin(5*t), t=0..20*Pi], thickness=3);

plotsetup(gif, plotoutput='38plot4.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
plot([[cos(t), sin(t), t=0..2*Pi],
[cos(t)+sin(5*t)/10, sin(t)+cos(5*t)/10, t=0..2*Pi]], thickness=3);

plotsetup(gif, plotoutput='38plot5.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
spacecurve([cos(t), sin(t), t], t=0..4*Pi, axes=box, thickness=3);

plotsetup(gif, plotoutput='38plot6.gif',
plotoptions='portrait, noborder, height=1000, width=1000');
spacecurve([sin(2*t), sin(5*t), t], t=0..2*Pi, axes=box, thickness=3);
Figure 1. Unit Circle, $r = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. 
Figure 2. Periodic Orbit, \( \mathbf{r} = \langle \sin 2t, \sin 3t \rangle \), \( 0 \leq t \leq 2\pi \).
Figure 3. Another Periodic Orbit, \( r = \langle \sin 2t, \sin 5t \rangle, \ 0 \leq t \leq 2\pi \).
Figure 4. Aperiodic Orbit, $r = \langle \sin \pi t, \sin 5t \rangle$, $0 \leq t \leq 20\pi$. 

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Figure 5. Planet and Moon, $\mathbf{r} = \langle \cos t + \frac{\sin 5t}{10}, \sin t + \frac{\cos 5t}{10} \rangle$, $0 \leq t \leq 2\pi$. 
Figure 6. Helix, $\mathbf{r} = \langle \cos t, \sin t, t \rangle$, $0 \leq t \leq 2\pi$. 
Figure 7. Space Curve, $\mathbf{r} = < \cos 2t, \sin 5t, t >$, $0 \leq t \leq 2\pi$. 
Derivatives and Integrals

Velocity and Acceleration

\[ \mathbf{r}'(t) = \lim_{{h \to 0}} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{{h \to 0}} \frac{\langle f(t+h), g(t+h) - g(t) \rangle}{h} \]

\[ = \lim_{{h \to 0}} \langle \frac{f(t+h)-f(t)}{h}, \frac{g(t+h)-g(t)}{h} \rangle \]

\[ = \langle \lim_{{h \to 0}} f(t), \lim_{{h \to 0}} g(t) \rangle \]

\[ = \langle f'(t), g'(t) \rangle \]