9.5 Lines and Planes

- Will draw vectors in the plane, but everything works in 3-space.

- **Vector Equation of a line:**

\[
r = r_0 + tv
\]

- \( r \) is the terminal point of a vector from the origin to a point on the line.
- \( r_0 \) is a particular such point.
- \( t \) is a parameter. It can assume all real values.
- \( v \) is a vector in the plane.
\[ \mathbf{r} = \mathbf{r}_0 + \ell \mathbf{v} \]

- Suppose \( \mathbf{r} = \langle x, y, z \rangle, \quad \mathbf{v} = \langle a, b, c \rangle \quad \text{and} \quad \mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle \)

- Then (1) turns into the **parametric equation**

\[
\begin{align*}
x &= x_0 + at, \\
y &= y_0 + bt, \\
z &= z_0 + ct
\end{align*}
\]

- Example: The line that passes through the point \( (5, 1, 3) \) and is parallel to the vector \( \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \).

\[ \langle x, y, z \rangle = \langle 5, 1, 3 \rangle + t \langle 1, 4, -2 \rangle \]
• We can eliminate $t$ in

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \]

to get the **symmetric equations**: 

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

(supposing $a$, $b$, and $c$ are all non-zero.)
Example 2: Find an equation for the line through the points $A = (2, 4, -3)$ and $B = (3, -1, 1)$.

\[
\mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{v} \\
= \langle 2, 4, -3 \rangle + t \langle 1, 5, 9 \rangle \\
= \langle 2, 4, -3 \rangle + d \frac{\langle 1, 5, 9 \rangle}{\sqrt{42}}
\]

$|d|$ = distance between $\mathbf{r}_0$ and $\mathbf{r}(t)$
- Vector Equations for a line segment from $r_0$ to $r_1$:

$$r(t) = (1 - t)r_0 + tr_1 = r_0 + t(r_1 - r_0), \quad 0 \leq t \leq 1$$

$0 \leq t \leq 1$
Planes

- A plane in three-dimensional space is determined by a point in the plane and a vector orthogonal to the plane. In this context, that vector is called the normal vector, or simply the normal of that plane.

- Example 4: Find an equation of the plane through the point \((2, 4, -1)\) with normal vector \(<2, 3, 4>\).

\[
\vec{r}(t) = \vec{r}_0 + s \vec{v} + t \vec{w}
\]

\[
\langle x-2, y-4, z+1 \rangle \cdot <2, 3, 4> = 0
\]

\[
2(x-2) + 3(y-4) + 4(z+1) = 0
\]

\[
2x + 3y + 4z - 12 = 0
\]

\[
2x + 3y + 4z = 12
\]

\(<2, 3, 4> \text{ normal vector}
\]

\[
a x + b y + c z + d = 0
\]

\(<a, b, c> \text{ normal vector}
• In general, find an equation of the plane containing the point \((x_0, y_0, z_0)\) and normal vector \(\mathbf{n} = \langle a, b, c \rangle\).

\[
\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0
\]

\[
a x + b y + c z - \left( a x_0 + b y_0 + c z_0 \right) = 0
\]

\[
d
\]
• Example 5:

Find an equation for the plane that passes through the points

\[ P = (1, 3, 2), \quad Q = (3, -1, 6), \quad \text{and} \quad R = (5, 2, 0). \]

\[ \langle x_0, y_0, z_0 \rangle = \langle 1, 3, 2 \rangle \]

\[ \mathbf{Q} - \mathbf{P}: \quad \langle 2, -4, 4 \rangle = \mathbf{v} \]

\[ \mathbf{R} - \mathbf{P}: \quad \langle 4, -1, -2 \rangle = \mathbf{w} \]

\[ \mathbf{n} = \mathbf{v} \times \mathbf{w} \]

\[
\begin{vmatrix}
  i & j & k \\
  2 & -4 & 4 \\
  4 & -1 & -2
\end{vmatrix} = \langle 12, 20, 14 \rangle
\]

\[ \langle x - 1, y - 3, z - 2 \rangle \cdot \langle 12, 20, 14 \rangle = 0 \]

\[ 12x + 20y + 14z = 100 \]
Example 7: Find the angle between the planes

\[ x + y + z = 1 \quad \text{and} \quad x - 2y + 3z = 1 \]

and an equation for the line of intersection.

\[
\mathbf{n}_1 = \langle 1, 1, 1 \rangle \\
\mathbf{n}_2 = \langle 1, -2, 3 \rangle \\
\mathbf{a} \cdot \mathbf{b} = 1 \cdot (1 + 3 + 3) = 7 \\
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{1 \cdot 1 \cdot 1} \\
\theta = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{1 \cdot 1 \cdot 1} \\
\theta = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{1} \sqrt{1} \sqrt{1}} \\
\approx 72^\circ
\]
Example 8. Find a formula for the distance $D$ of the point $P_1 = (x_1, y_1, z_1)$ from the plane

$$ax + by + cz + d = 0.$$ 

- The distance is the length of the projection of a vector from a point in the plane to $P_1$ onto the normal vector $\langle a, b, c \rangle$.

- Recall

$$\text{proj}_a b = \frac{b \cdot a}{a \cdot a} a = \frac{b \cdot a}{|a||a|} a.$$

- $\frac{a}{|a|}$ is a unit vector and $\frac{b \cdot a}{|a|}$ is the **scalar projection** of $b$ onto $a$.

- The distance of $P_1$ from the plane is the absolute value of the scalar projection of a vector from a point in the plane to $P_1$. See Figure 12 on page 669 of the textbook.

- Express it in terms of a general point $(x_0, y_0, z_0)$ in the plane.
• Example 9. Computing the distance between two parallel planes.
• Computing the distance between two skew lines. (skew means the lines are not parallel and do not intersect).