Review

- vectors, vector addition, scalar multiplication
- The length of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is
  
  $$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$ 

- Equivalently
  
  $$|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2.$$ 

9.3 The Dot Product

- The key concept today: Suppose
  
  $$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \quad \text{and} \quad \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

  Then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

  $$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$ 

- Similarly in 2 or $n$ dimensional space.
- Major fact: If $\theta$ is the angle formed by $\mathbf{a}$ and $\mathbf{b}$ then
  
  $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$ 

- Also note that
  
  $$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}.$$
• Examples:

\[ \langle 1, 2 \rangle \cdot \langle -2, 1 \rangle = 0 \]

\[ \langle 5, 2 \rangle = \alpha \]

\[ \langle 8, -2 \rangle = \beta \]

\[ \alpha \cdot \beta = 36 \]

\textbf{Figure 1.} The Dot Product.
A Physical Motivation

- work, pulling at an angle.

\[
\text{Effective force} = \frac{|E|}{|F|} = \cos \theta \\
|E| = |F| \cos \theta \\
\text{Work} = s |E| = s |F| \cos \theta = d \cdot F
\]

**Figure 2.** Pulling at an Angle.
Notes $\langle 1, 1 \rangle \cdot \langle -2, 1 \rangle = -1$

- The **dot product** is so called because of the dot in $a \cdot b$. (Of course).
- It is also called the **scalar product**, or **inner product**.
- The dot product of two vectors is a scalar, not a vector!
- The dot product of two vectors may be negative!

- The dot product of two vectors is zero if and only if the two vectors form a right angle. In that case the vectors are called **perpendicular**, **orthogonal**, or, sometimes, **normal**.
- Qualitatively, the dot product $a \cdot b$ measures the degree to which the two vectors $a$ and $b$ point in the same direction.
- The zero vector is orthogonal to all other vectors.
Properties of the Dot Product

- Except for number 6., the following properties can be verified easily by going back to the definition of the dot product.

1. \( a \cdot a = |a|^2 \)
2. \( a \cdot b = b \cdot a \)
3. \( a \cdot (b + c) = a \cdot b + a \cdot c \)
4. \( (ca) \cdot b = c(a \cdot b) = a \cdot (cb) \)
5. \( 0 \cdot a = 0 \)
6. \( a \cdot b = 0 \iff a \perp b \)
Algebraic Form of the Dot Product

- Recall the Law of Cosines:

\[ c^2 = a^2 + b^2 - 2ab \cos(\gamma) \]

**Figure 3.** The Law of Cosines.


- The law of cosines is a generalization of the Law of Pythagoras.
\[(\beta - \xi)^2 = \beta^2 - 2\beta \xi + \xi^2\]

- Apply the Law of Cosines to the triangle with sides \(a\), \(b\), and \(a - b\)

\[b = a + b - a\]

\[|b|^2 = |b_1, b_2|^2 = b_1^2 + b_2^2\]

\[|b-a|^2 = (b-a) \cdot (b-a)\]

\[= b \cdot b + a \cdot a - 2(a \cdot b) \quad \text{LOC}\]

\[= |b|^2 + |a|^2 - 2|a||b|\cos \theta\]

\[= b \cdot b + a \cdot a - 2|a||b|\cos \theta\]

\(\Rightarrow \quad a \cdot b = |a||b|\cos \theta\)
Examples

$$\theta = 45^\circ$$

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\begin{align*}
\langle 1, 1 \rangle & \cdot \langle 1, 1 \rangle = \sqrt{2} \\
\langle 1, 0 \rangle & \cdot \langle 1, 0 \rangle = 1
\end{align*}$$

$$|\langle 1, 1 \rangle| \cdot |\langle 1, 0 \rangle| \cdot \cos \theta = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1$$

$$\langle 1, 1 \rangle \cdot \langle 1, 0 \rangle = 1$$

$$a = \langle 2, 3 \rangle \quad b = \langle 8, 9 \rangle$$
\[
\text{proj}_a b = \frac{a \cdot b}{a \cdot a} a = \frac{43}{145} \langle 8, 9 \rangle
\]

\[
a \cdot b = |a| |b| \cos \theta
\]

\[
\cos \theta = \frac{a \cdot b}{|a| |b|} = \frac{43}{\sqrt{13} \sqrt{145}}
\]

\[
\theta = \arccos \left( \frac{43}{\sqrt{13} \sqrt{145}} \right)
\]
Orthogonal Projection

- The textbook distinguishes a scalar projection and a vector projection. Usually projection, or orthogonal projection, means vector projection.

![Diagram of orthogonal projection](http://www.physics.brocku.ca/PPLATO/h-flap/math2_6f_4.png)

**Figure 4.** Projections.


\[
\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{|a||b|\cos \theta}{|a|} \quad \text{scalar projection of } b \text{ onto } a
\]

\[
\text{proj}_a b = \frac{a \cdot b}{|a|^2} a = \frac{a \cdot b}{|a|^2} a \quad \text{vector projection of } b \text{ onto } a
\]
Examples

Figure 5. Projections.
\[ b = \langle 3, 4 \rangle \]

\[ a = \langle 8, -1 \rangle \]

\[ p = \text{proj}_a b = \frac{a \cdot b}{a \cdot a} a = \frac{20}{65} \langle 8, -1 \rangle \approx \langle 2.46, -0.31 \rangle \]

**Figure 6.** Projections.
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\begin{align*}
\text{planet} & : \quad 3.4 \times 10^3 < \cos \frac{2\pi t}{5}, \sin \frac{2\pi t}{5} > \\
\text{moon} & : \quad \left< \cos \left(18\pi t + \frac{\pi}{2}\right), \sin \left(18\pi t + \frac{\pi}{2}\right) \right>
\end{align*}

\begin{align*}
\mathbf{x}(t) & = 3.4 \cdot 10^3 \cos \frac{2\pi t}{5} + \cos \left(18\pi t + \frac{\pi}{2}\right) \\
\mathbf{y}(t) & = 3.4 \cdot 10^3 \sin \frac{2\pi t}{5} + \sin \left(18\pi t + \frac{\pi}{2}\right)
\end{align*}