9. 3-space

9.1 Three-dimensional Coordinate system

- Basic idea: like 2-dimensional system, except we also have a z axis which is perpendicular to the $xy$-plane.

- Describe the location of a point by how far to go north, east, and up.

![3D Coordinate System](http://www.shelovesmath.com/wp-content/uploads/2015/05/3D-Coordinates-1.jpg)

**Figure 1.** 3-space.

- $x,y,z$-axis, $xy,xz,yz$-plane
- 8 octants (corresponding to four quadrants in 2D).
Right Hand System

- if you are looking along the $z$-axis towards the $x-y$ plane so that it appears as we usually draw the $x-y$ plane, does the $z$-axis point towards you or away from you?  
- It points towards you.  
- That’s a convention, not a mathematical necessity. Math would work the other way as well. The system is called right handed.

![Diagram of Right Hand System](https://share.ehs.uen.org/node/8523)

**Figure 2.** Right Hand System.

- from [https://share.ehs.uen.org/node/8523](https://share.ehs.uen.org/node/8523)
• Projections into the planes and axes. Set the missing coordinates to zero.

• works in 2D and 3D.

\[(1, 2, 3) \Rightarrow (1, 0, 3) \quad \text{\(x\)-\(z\) plane}\]

\[\Rightarrow (0, 2, 0) \quad \text{\(y\)-axis}\]
• The **graph** of an equation (in the variables \(x, y, z\)) is the set of all points in 3-space whose coordinates satisfy the equation.

• Examples

  • \(x = 0\)  
    \[ (0, y, z) \quad \text{Y-z plane} \]

  • \(z = 0\)  
    \[ x-y \text{ plane} \]

  • \(z = 3\)  
    parallel to the \(x-y\) plane
    contains \((0,0,3)\)

  • \(x^2 + y^2 = 1\)  
    cylinder \(r = 1\)
    \[ \text{axis is the Z-axis} \]

  • \(x^2 + y^2 = 0\)  
    Z-axis
\begin{itemize}
  \item $3 = 1 + 2$
  \item $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$
  \item $3 = 2 + 2$
\end{itemize}

empty set
Distance between points \((x_1, y_1)\) and \((x_2, y_2)\)

\[
d_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}
\]

Distance between points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\)

\[
d_3 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}
\]
\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

\[ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
Equation of a sphere

\[(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2\]

- Example 6: Find the center and radius of the sphere

\[x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0.\]

\[x^2 + y^2 + 6x + 8y = 0\]

\[x^2 + 6x + 9 + y^2 + 8y + 16 = 9 + 16\]

\[(x + 3)^2 + (y + 4)^2 = 25 = r^2\]

\[r = 5\]

\[(h, k) = (-3, -4)\]

\[x^2 + 4x + y + y^2 - 6y + 9 + z^2 + 2z + 1 = 4 + 9 + 1 - 6\]

\[(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8 = r^2\]

Center \((-2, 3, -1)\) \[r = \sqrt{8}\]
Problem 20, page 639. Find the equation of the largest sphere with center (5,4,9) that is contained in the first octant (where \( x, y, z \geq 0 \)).

\[
(x - 5)^2 + (y - 4)^2 + (z - 9)^2 = 16
\]
• What is the graph of the inequalities

\[ 1 \leq x^2 + y^2 \leq 4 \quad \text{and} \quad 0 \leq z \leq 3 \]
• Problem 41, page 639. Find the distance between the spheres

\[ x^2 + y^2 + z^2 = 4 \quad \text{and} \quad x^2 + y^2 + z^2 = 4x + 4y + 4z - 11. \]

\[
\begin{align*}
\mathbf{r} &= \mathbf{c} \\
&= \mathbf{z} \\
(\mathbf{x} - 2) + (\mathbf{y} - 2) + (\mathbf{z} - 2)^2 &= -11 + 4 + 4 + 4 \\
\mathbf{r} &= 1 \\
d \left( (0,0,0), (2,2,2) \right) &= \sqrt{127} \\
d &= \sqrt{127} - 2 - 1
\end{align*}
\]
• **Vectors** have length and direction

• ** Scalars** have size only.
\[ e^x + e^{2x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \]

\[ x^n(2x)^n = x^n + 2^n x^n \]

\[ = (1 + 2^n)x^n \]

\[ f(\omega) = 2 \]
\[ f'(\omega) = 3 \]
\[ f''(\omega) = 5 \]
\[ f'''(\omega) = 10 \]