8.7 Taylor Series

- **Taylor Series** of the function $f$ at $a$ (or about $a$, or centered at $a$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

$$= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots$$

- In the special case that $a = 0$ this is also called a **MacLaurin Series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

- **Brook Taylor**, 1685-1731.
- **Colin MacLaurin**, 1698-1746.
- Basic principle: match derivatives

$$\frac{d^k}{dx^k} \left[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right]_{x=0} = f^{(k)}(0)$$

because

$$\frac{d^k}{dx^k} x^n \bigg|_{x=0} = \begin{cases} k! & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$
Problem 68, page 618. Compute the MacLaurin Series of

\[ f(x) = e^{-\frac{1}{x^2}} \]

\[ f(0) = 0 \]

\[ y = \frac{1}{x^2} \]

\[ z = \frac{1}{x^2} \]

\[ \lim_{x \to 0} e^{-1/x^2} = \lim_{z \to 0} e^{-z} = 0 \]

\[ f(x) = \begin{cases} 
  e^{-1/x^2} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \]

\[ e^{-z} = \frac{1}{e^z} \]
\[ f(x) = e^{-\frac{1}{2}x^2} \]

\[ f'(x) = e^{-\frac{1}{2}x^2} \frac{d}{dx} \left( -\frac{1}{x^2} \right) \]

\[ = -e^{-\frac{1}{2}x^2} \frac{d}{dx} x^{-2} \]

\[ = -2x^{-3}e^{-\frac{1}{2}x^2} = \frac{e^{-\frac{1}{2}x^2}}{x^3} = \frac{e^{-2}}{2^{3/2}} \]

\[ f(0) = 0 \]

\[ f^{(n)}(0) \]
• We need to compute \( L = \lim_{x \to 0} e^{-1/x^2} \). With the substitution
\[
z = 1/x^2
\]
this turns into
\[
L = \lim_{x \to 0} e^{-1/x^2} = \lim_{z \to \infty} e^{-z} = \lim_{z \to \infty} \frac{1}{e^z} = 0.
\]

• Notice that all derivatives of \( e^{-z} \) also go to zero as \( z \) goes to infinity. That’s highly suggestive but not quite enough to show that all derivatives of \( f(x) \) go to zero as \( x \) goes to zero.

Think about why it’s not enough.

• Consider the derivative of \( f \). We get
\[
f'(x) = \frac{d}{dx} e^{-1/x^2} = 2x^{-3}e^{-1/x^2} = \frac{2z^{3/2}}{e^z}.
\]

• We consider the case that \( z \) goes to infinity. Both numerator and denominator go to infinity. We could apply the rule of L’Hopital. But notice that we have an exponential in the denominator and a square root in the numerator. The exponential goes to infinity faster than any polynomial which in turn goes faster than the square root. Thus
\[
f'(0) = \lim_{z \to \infty} \frac{2z^{3/2}}{e^z} = 0.
\]

• Consider the second derivative. We apply the product rule to \( f' \):
\[
f''(x) = \frac{d}{dx} 2x^{-3}e^{-1/x^2} = -6x^{-4}e^{-1/x^2} + (2x^{-3})^2 e^{-1/x^2} = \frac{-6z^2 + 2z^{3/2}}{e^z}.
\]

• Again,
\[
f''(0) = \lim_{x \to 0} f''(x) = 0.
\]
Now note that the pattern repeats! Differentiating again, using the product rule never gets rid of the factor $e^{-1/x^2} = e^{-z} = 1/e^z$. We multiply that factor with polynomials and square roots in $z$. The product goes to zero as $z$ goes to infinity.

**all** derivatives of $f$ are zero at zero!!

The MacLaurin Series of $f$ is zero! It converges only at $x = 0$. 
Figure 1. Graph of $f(x) = e^{-1/x^2}$. 
Figure 2. Graph of $f(x) = e^{-1/x^2}$.
Equation Solving

Reminder of Newton’s Method

- Section 4.7, discussed in Math 1310.
- It produces convergent or divergent sequences.
- Want to solve $f(x) = 0$.
- Local Linearization, major idea.

$$x_0 \text{ given } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Figure 3. Newton’s Method.

- Figure 3 from
  http://tutorial.math.lamar.edu/Classes/CalcI/NewtonsMethod.aspx

- Isaac Newton, 1643–1727, invented Calculus, among many other things.
• Let’s take a different look at Newton’s Method.

• Suppose we want to solve

\[ f(x) = 0, \quad (1) \]

as before. We don’t know the solution, but let’s give it a name, let’s call it \( \alpha \).

• Now suppose \( F \) is the inverse function of \( f \), i.e.,

\[ F(f(x)) = x. \quad (2) \]

Applying the inverse function on both sides of (1) gives

\[ F(f(x)) = F(0) \implies x = F(0). \quad (3) \]

• Also suppose we have an approximation \( x_n \) of the root of \( f \). Let

\[ y_n = f(x_n) \quad \text{and} \quad x_n = F(y_n). \]

How about approximating \( F(0) \) in (3) by a Taylor Series centered at \( y_n \)? We get

\[ \alpha = F(0) \approx F(y_n) + F'(y_n)(0 - y_n). \quad (4) \]

• OK, but what’s \( F'(y_n) \)?

• Differentiating in (2) gives

\[ F'(f(x))f'(x) = 1 \]

which implies

\[ F'(f(x)) = \frac{1}{f'(x)}. \quad (5) \]

• (We did compute that differentiation formula for the inverse function in Math 1310.)
• Substituting $y_n = f(x_n)$ and $F'(y_n) = 1/f'(x_n)$ in (4) gives

$$\alpha \approx x_n + 1 = F(y_n) - F'(y_n)y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (6)$$

• That’s Newton’s Method!

![This makes sense, we approximate the function by its tangent, which is a first order Taylor series.]

• OK, how about using a higher order Taylor Series?

$$\alpha = F(0) \approx F(y_n) + F'(y_n)(0 - y_n) + \frac{1}{2} F''(y_n)(0 - y_n)^2. \quad (7)$$

• So what’s that second order derivative?

• Differentiating in (5) gives

$$F''(f(x))f'(x) = -\frac{f''(x)}{(f'(x))^2},$$

i.e.,

$$F''(f(x)) = -\frac{f''(x)}{(f'(x))^3}.$$

• Substituting in (7) gives the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)(f(x_n))^2}{2(f'(x_n))^3}. \quad (8)$$

• So how does that method compare with Newton’s Method?

• To illustrate the behavior let’s suppose we want to approximate the square root of 2 by solving

$$f(x) = x^2 - 2 = 0,$$
starting at $x_0 = 1$.

1 restart;
2 Digits:=200;
3 f:=x**2-2;
4 sol:=sqrt(2);
5 s:=1;
6 fp:=diff(f,x);
7 fpp:=diff(fp,x);
8 z:=x-f/fp-fpp*f**2/2/fp**3;
9
10 s:=evalf(subs(x=s,z));evalf(sol-s);
11 s:=evalf(subs(x=s,z));evalf(sol-s);
12 s:=evalf(subs(x=s,z));evalf(sol-s);
13 s:=evalf(subs(x=s,z));evalf(sol-s);
14 s:=evalf(subs(x=s,z));evalf(sol-s);
15 s:=evalf(subs(x=s,z));evalf(sol-s);
16
17 s:=1;
18
19 z:=x-f/fp;
20
21
22 s:=evalf(subs(x=s,z));evalf(sol-s);
23 s:=evalf(subs(x=s,z));evalf(sol-s);
24 s:=evalf(subs(x=s,z));evalf(sol-s);
25 s:=evalf(subs(x=s,z));evalf(sol-s);
26 s:=evalf(subs(x=s,z));evalf(sol-s);
27 s:=evalf(subs(x=s,z));evalf(sol-s);
28 s:=evalf(subs(x=s,z));evalf(sol-s);
29 s:=evalf(subs(x=s,z));evalf(sol-s);
30 s:=evalf(subs(x=s,z));evalf(sol-s);
31
32
33

- The following Table lists the errors, $\sqrt{2} - x_n$, for both methods. Remember that a number like 0.39E-1 means $0.39 \times 10^{-1} = 0.039$. The fifth step of the method (8) gives an approximation that is accurate to approximately 137 digits. The next iterate would be accurate to more than 400 digits, but the arithmetic in this case carries only 200 decimal digits.
<table>
<thead>
<tr>
<th>(n)</th>
<th>Newton</th>
<th>Order 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-0.86E - 1)</td>
<td>(0.39E - 1)</td>
</tr>
<tr>
<td>2</td>
<td>(-0.25E - 2)</td>
<td>(0.16E - 4)</td>
</tr>
<tr>
<td>3</td>
<td>(-0.21E - 5)</td>
<td>(0.10E - 14)</td>
</tr>
<tr>
<td>4</td>
<td>(-0.15E - 11)</td>
<td>(0.28E - 45)</td>
</tr>
<tr>
<td>5</td>
<td>(-0.90E - 24)</td>
<td>(0.54E - 137)</td>
</tr>
<tr>
<td>6</td>
<td>(-0.29E - 48)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(-0.29E - 97)</td>
<td></td>
</tr>
</tbody>
</table>

- Clearly, not withstanding the impressive performance of Newton’s Method, the second method (8) performs significantly better. Roughly speaking, Newton’s Method doubles the number of correct decimal digits at every step, while the second method triples it.
- Of course, the second method does require the computation of the second derivative of \(f\).
- **Exercise:** Explore using a third order Taylor series for \(F\).
- This completes our regular discussion of Taylor Series.
- Next week on Friday we will have our first exam. There may be a problem or two from chapter 6, but the main focus of the exam will be on Chapter 8.
- On Monday we’ll start our new subject. On Tuesday, Wednesday, and Thursday in the Lab, we will have a review of chapter 8.
- Any questions?
A Challenge

- email me your answer to the following puzzle by next Friday. It will exercise your three dimensional thinking! I will forward the clearest answer(s) to the whole class with your name attached.

- Consider two lines $L$ and $M$ in three dimensional space. Assume the lines do not intersect and are not parallel. (Such lines are said to be in general position.) A connecting line is a line that intersects both $L$ and $M$. Let $S$ be the set of all points in three dimensional space that lie on some connecting line. (In other words, a point $P$ is in $S$ if there exists a line that intersects both $L$ and $M$ and contains $P$.) In plain English, describe the shape of that set $S$!
\[
\frac{1}{2 + r} = \frac{1}{2} \left( \frac{1}{1 + \frac{x}{a}} \right) = \frac{1}{2} \cdot \frac{1}{1 - r} \quad r = -\frac{x}{2}
\]

\[
= \frac{1}{2} \left( 1 + r + r^2 + \ldots \right)
\]

\[
= \frac{1}{2} \left( 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} \frac{x^k}{2^k} (-1)^k + \frac{1}{2}
\]

\[
= \frac{1}{2} x + \sum_{k=1}^{\infty} \frac{x^{k+1}}{(k+1)2^{k+1}} (-1)^k
\]

\[
= \frac{1}{2} x + \sum_{k=2}^{\infty} \frac{x^k}{k2^k} (-1)^{k-1}
\]

\[
= \sum_{k=1}^{\infty} \frac{x^k}{k2^k} (-1)^{k+1}
\]

\[+ \frac{1}{2}
\]
\[ f(x) = \ln(x+2) \quad f(0) = \ln 2 \]
\[ f'(x) = \frac{1}{x+2} = (x+2)^{-1} \quad f'(0) = \frac{1}{2} \]
\[ f^{(n)} = (n-1)! \cdot (-1)^{n+1} \cdot \frac{1}{2^n} \]
\[ \sum \frac{(n-1)! \cdot (-1)^{n+1}}{2^n \cdot n!} x^n \]
\[ \sum \frac{(-1)^{n+1} \cdot x^n}{2^n \cdot n!} \]