8.7 Taylor Series

- **Taylor Series** of the function $f$ at $a$ (or about $a$, or centered at $a$):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

\[
= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots
\]

- In the special case that $a = 0$ this is also called a **MacLaurin Series**:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n
\]

\[
= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots
\]

- **Brook Taylor**, 1685-1731.
- **Colin MacLaurin**, 1698-1746.

- Basic principle: match derivatives

\[
\left. \frac{d^k}{dx^k} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right|_{x=0} = f^{(k)}(0)
\]

because

\[
\left. \frac{d^k}{dx^k} x^n \right|_{x=0} = \begin{cases} 
  k! & \text{if } n = k \\
  0 & \text{if } n \neq k 
\end{cases}
\]
Let’s look more closely at the quality of the polynomial approximation.

\[ f(x) = T_n(x) + R_n(x) \]

where the \( n \)-degree Taylor polynomial for \( f \) about \( a \) is

\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k
\]

\[
= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\]

and

\[ R_n(x) = f(x) - T_n(x) \]

is the remainder.

• The remainder can be written in several different ways.

• The integral form is

\[
R_n(x) = \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t)dt
\]

and

• Lagrange’s form says that there exists a number \( z \) between \( a \) and \( x \) such that

\[
R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}
\]

Thus the remainder looks exactly like the next term in the Taylor series, except that the derivative is evaluated at an unknown point which we know to exist.

• Immediate consequences: If the derivatives are bounded the Remainder converges to zero. For example, the Taylor series for the exponential, sin, and cos all have an infinite radius of convergence.
Derivation of Taylor’s formula

• Recall integration by parts:

\[
\int_a^b u'(t)v(t)\,dt = u(t)v(t)\bigg|_a^b - \int_a^b u(t)v'(t)\,dt.
\]

• We get

\[
f(x) = f(0) + f(x) - f(0) \\
= f(0) + \int_0^x f'(t)\,dt \\
= f(0) + \int_0^x 1 \times f'(t)\,dt
\]

integration by parts, \( u'(t) = 1, u(t) = t - x, v(t) = f'(t) \)

\[
= f(0) + (t - x)f'(t)\bigg|_0^x - \int_0^x (t - x)f''(t)\,dt
\]

integration by parts again ...

\[
= f(0) + xf'(0) - \left[\frac{1}{2}(t - x)^2 f''(t)\right]_0^x + \int_0^x \frac{1}{2} (t - x)^2 f'''(t)\,dt
\]

and again ...

\[
= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \left[\frac{1}{6}(t - x)^3 f'''(t)\right]_0^x - \int_0^x \frac{1}{2}(t - x)^3 f''''(t)\,dt
\]

...leading to:

\[
= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + R_n(x)
\]

where

\[
R_n(x) = \frac{(-1)^n}{n!} \int_0^x (t-x)^n f^{(n+1)}(t)\,dt = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)\,dt.
\]
8.8 Applications

• Example: Approximate the exponential in the interval \([-1,1]\) by a Taylor polynomial up to a certain accuracy.

\[
e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + R_n(x)
\]

where

\[
R_n(x) = \frac{x^{n+1}}{(n+1)!}e^{z(x)}
\]

where

\[-1 \leq z(x) \leq 1\]

• Thus

\[e^{-1} \leq e^z \leq e \quad \text{and} \quad |R_n(x)| \leq \frac{e}{(n+1)!} = E\]

The following table shows \(E\) for some values of \(n\):

<table>
<thead>
<tr>
<th>(n)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E)</td>
<td>(3.7 \times 10^{-3})</td>
<td>(5.8 \times 10^{-7})</td>
<td>(1.3 \times 10^{-13})</td>
<td>(5.3 \times 10^{-20})</td>
<td>(6.7 \times 10^{-27})</td>
<td>(3.3 \times 10^{-34})</td>
</tr>
</tbody>
</table>

• A typical floating point system carries about 16 digits. So you could evaluate the exponential in the interval \([-1,1]\) to the accuracy that can be expressed in that system using a Taylor polynomial of degree 18 for which \(E\) would be \(2.2 \times 10^{-17}\).

• By the same token you could evaluate \(\sin\) and \(\cos\) in that interval to a slightly higher accuracy since the maximum value of the derivatives in that interval is 1.
• Example 3, textbook. Consider an object that when at rest has a mass $m_0$.

• Recall Einstein’s famous equation:

\[ E = m_0 c^2 \]

where $c = 299,792,458 \text{ m/s}$ is the speed of light (about 186,000 miles per second).

• That’s the energy equivalent to a mass $m_0$ at rest.

• As an object moves it gets more massive. Its mass at velocity $v$ is given by

\[ m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \]

• The kinetic energy of an object moving at velocity $v$ is the difference between its total energy and its energy at rest:

\[ K = mc^2 - m_0 c^2 \]

• In past calculations (for example the calculation of escape velocity) we have used $K = \frac{1}{2}m_0 v^2$. How is that consistent with the theory of relativity?

• Let’s expand $K$ into a MacLaurin series in terms of

\[ x = \frac{v^2}{c^2}. \]

• We get

\[ K = K(x) = mc^2 - m_0 c^2 = \left( \frac{m_0}{\sqrt{1 - x}} - m_0 \right) c^2 = \left( \frac{1}{\sqrt{1 - x}} - 1 \right) m_0 c^2 \]
• Compute the first few terms of the MacLaurin series of

\[
f(x) = \frac{1}{\sqrt{1 - x}} - 1
\]

\[
f'(x) = \frac{d}{dx} (1-x)^{-\frac{1}{2}} = \frac{1}{2} (1-x)^{-\frac{3}{2}}
\]

\[
f''(x) = \frac{3}{4} (1-x)^{-\frac{5}{2}}
\]

\[
f'''(x) = \frac{15}{8} (1-x)^{-\frac{7}{2}}
\]

\[
f(x) = \frac{x}{2} + \frac{3}{4} \cdot \frac{x^2}{2!} + \frac{15}{8} \cdot \frac{x^3}{3!} + \ldots
\]

\[
= \frac{x}{2} + \frac{3}{8} x^2 + \frac{5}{16} x^3 + \ldots
\]
• We have

\[ f(x) = 0 + \frac{x}{2} + \frac{3x^2}{8} + \frac{15x^3}{48} + \ldots \]

and

\[ K(x) = f(x)m_0c^2 = \left( \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \ldots \right)m_0c^2 \]

• At ordinary speeds \( v \) the term \( x = \frac{v^2}{c^2} \) is very small. Ignoring the higher order terms we get

\[ K \approx \frac{x}{2}m_0c^2 = \frac{1}{2}m_0v^2 \]

• Thus for small speeds the theory of relativity is consistent with Newtonian mechanics.

• How big is the error?

• Consider the first ignored term in the Taylor series compared to the leading term:

• We get

\[ \frac{f(x) - x/2}{x/2} \approx \frac{3x^2}{8} = \frac{3x}{4} = R. \]

• A more accurate comparison of course is obtained by evaluating the relative error

\[ T = \frac{f(x) - x/2}{f(x)} \]

directly.

• The following table gives the values of that ratio for some speeds, measured in miles per second.
The relative error in the Newtonian analysis is small even for speeds that are large by our earthly experience. Ignoring relativity affects the relative error only in its 11-th digit at a speed of 3600 miles per hour—faster than any airplane—and only the 8 place at a speed of 10 miles per second, faster than the escape velocity of earth.

The error estimate via the next term in the Taylor series is very accurate up to even higher speeds.
Some Important MacLaurin Series

- The following Table is from page 613 of the textbook

<table>
<thead>
<tr>
<th>Function</th>
<th>Series</th>
<th>Radius of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{1-x}$</td>
<td>$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$</td>
<td>$R = 1$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$</td>
<td>$R = \infty$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$</td>
<td>$R = \infty$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$</td>
<td>$R = \infty$</td>
</tr>
<tr>
<td>$\text{arctan } x$</td>
<td>$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1!} = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots$</td>
<td>$R = 1$</td>
</tr>
<tr>
<td>$\ln(1 + x)$</td>
<td>$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$</td>
<td>$R = 1$</td>
</tr>
<tr>
<td>$(1 + x)^k$</td>
<td>$\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)x^2}{2!} + \ldots$</td>
<td>$R = 1$</td>
</tr>
</tbody>
</table>

In the last formula, $k$ is any real number and

$$\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$$
Complex Numbers

- Remember complex numbers:

\[ z = a + bi \quad \text{where} \quad i^2 = -1. \]

- The following table lists the first few powers of \( i \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i^n )</td>
<td>1</td>
<td>( i )</td>
<td>-1</td>
<td>-( i )</td>
<td>1</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

- How about computing the MacLaurin series of \( e^z \)?
Q&A

#5

\[ f(x) = 1 + (x+2) + \frac{(x+2)^2}{\sqrt{2!}} \]

\[ = \sum_{k=0}^{\infty} \frac{(x+2)^k}{k!} \]

\[ R = \left( \frac{x+2}{\sqrt{k+1}} \right)^{k+1} \]

\[ = \left( \frac{x+2}{k+1} \right)^{k+1} \]

#3

\[ f(x) = x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 \]

\[ = \sum_{k=1}^{\infty} k^2 x^k \]

\[ R = \left( \frac{(k+1)^2 x^{k+1}}{k^2 x^k} \right) = \left( \frac{(k+1)^2}{k^2} \right) x \]
\[
\lim_{k \to \infty} \left( \frac{(k+1)^2}{k^2} x \right) = |x|
\]

abs. conv. if \( |x| < 1 \)

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1
\]

\# 8

\[
f(x) = 7 + 7x + 41x^2 + 41x^3 + \ldots
\]

\[
g(x) = 21 + 35x + 40x^2 + 62x^3 + \ldots
\]

\[
\frac{g(x)}{f(x)} = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots
\]

\[
(7 + 7x + 41x^2 + 41x^3 + \ldots) \left( c_0 + c_1x + c_2x^2 + \ldots \right) = 21 + 35x + \ldots
\]

\[
7c_0 + 7c_1x + 41c_2x^2 + 41c_3x^3 + \ldots
\]

\[
7c_1x + 7c_2x^2 + 41c_3x^3 + \ldots
\]

\[
7c_2x^2 + 7c_3x^3 + \ldots
\]

\[
7c_0 + 7(c_0 + c_1)x + \left( 41 + 7c_3 + c_2 \right)x^2 + \ldots
\]
\[ 7c_0 = 21 \quad c_0 = 3 \]
\[ 7(c_0 + c_1) = 35 \quad c_1 = 2 \]

\[ \sum_{k=1}^{\infty} \frac{2}{(2k+1)(2k+3)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \cdots \]

\[ \frac{2}{(2k+1)(2k+3)} = \frac{1}{2k+1} - \frac{1}{2k+3} \]

\[ \frac{2}{(2k+1)(2k+3)} = \frac{A}{2k+1} + \frac{B}{2k+3} \]

\[ A(2k+3) + B(2k+1) = 2 \]
\[ f(x) = 7 + 6x + 3x^2 + 4x^3 \]
\[ g(x) = 21 + 39x + 90x^2 + 67x^3 \]
\[ 7 + 6x + 3x^2 + 4x^3 \]
\[ \begin{align*}
&= 3 + 3x \\
&= 21 + 39x + 90x^2 + 67x^3 + \ldots \\
&= 21 + 18x + 9x^2 + 12x^3 \\
&= 21x + 39x^2 + 55x^3 \]
\]
\[(7 + 6x + 3x^2 + 4x^3)(c_0 + c_1x + c_2x^2 + \ldots)\]
\[7c_0 + 6c_0 + 3c_0 + c_2x + 4c_0 + 3c_0 + x^3\]
\[+ 7c_1x + 6c_1x^2 + 3c_1x^3\]
\[+ 3c_2x^2 + \ldots\]
\[= 7c_0 + (6c_0 + 7c_1)x + \ldots\]
\[7c_0 = 21 \quad c_0 = 3\]
\[ 6c_0 + 7c_1 = 39 = 18 + 7c_1, \quad c_1 = 3 \]

and so on

\[
\sum_{k=0}^{\infty} (k+1) \times k^3
\]

\[
\sum_{k=0}^{\infty} (k+1)(k+3) \times k^2
\]

\[
\sum_{k=0}^{\infty} (k+1)(k+3)(k+5) \times k
\]

\[
f(x) = 3 + 2x + 3x^2 + 3x^3 + \ldots
\]

\[
g(x) = 2 + 5x + 5x^2 + 2x^3 + \ldots
\]

\[
c + 4x + 6x^2 + 6x^3
\]

\[
15x + 10x^2 + 15x^3 + 15x^4
\]

\[
15x^2 + 10x^3 + \ldots
\]

\[
c + 19x + 31x^2 + \ldots
\]