8.7 Taylor Series

- **Taylor Series** of the function \( f \) at \( a \) (or about \( a \), or centered at \( a \)):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

\[
= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots
\]

- In the special case that \( a = 0 \) this is also called a **MacLaurin Series**:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

\[
= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots
\]

- The partial sums a Taylor Series are called **Taylor or MacLaurin Polynomials**.

- Basic principle: match derivatives

\[
\frac{d^k}{dx^k} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right) \bigg|_{x=0} = f^{(k)}(0)
\]

because

\[
\frac{d^k}{dx^k} x^n \bigg|_{x=0} = \begin{cases} 
  k! & \text{if } n = k \\
  0 & \text{if } n \neq k 
\end{cases}
\]
- MacLaurin Series for

\[ f(x) = 1 + x + 2x^2 - 3x^3 + x^4 \]

\[ = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \]

\[ f(x) = \frac{f^{(0)}(0)}{0!} x^0 = 1 \]

\[ f'(x) = 1 + 4x - 9x^2 + 4x^3 \]

\[ f'(x) = \frac{f^{(1)}(0)}{1!} x^1 = 1 \]

\[ f''(x) = 4 - (8x + 12x^2) \]

\[ f''(x) = \frac{f^{(2)}(0)}{2!} x^2 = 4 \]

\[ f'''(x) = -18 + 24x \]

\[ f'''(x) = \frac{f^{(3)}(0)}{3!} x^3 = -18 \]

\[ f^{(4)}(x) = 24 \]

\[ f^{(4)}(x) = \frac{f^{(4)}(0)}{4!} x^4 = 24 \]

0 ever after

\[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \]

\[ = 1 + x + 2x^2 - 3x^3 + x^4 \]

\[ a_n = \frac{f^{(n)}(0)}{n!} \]

\[ a_2 = 2 \]

\[ = 3 + 5(x-2) + 8(x-2)^2 + 5(x-2)^3 + 6(x-2)^4 \]

exercise: check
$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \ldots$

- **Example 11:** Obtain the MacLaurin Series for

$$f(x) = \int_0^x e^{-t^2} dt$$

$$f(x) = e^{-x^2}$$

$$f'(x) = -2x e^{-x^2}$$

$$f''(x) = -2 \left( e^{-x^2} + x e^{-x^2} \right) = -2 \left( 1 + x \right) e^{-x^2}$$

$$f'''(x) = -2 \left( e^{-x^2} + x e^{-x^2} + \frac{1}{2} x^2 e^{-x^2} \right)$$

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -2$$

$$f(x) = \int_0^x e^{-t^2} dt = \int_0^x 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} \ldots dt$$

$$= \left[ t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \ldots \right]$$
• Suppose $y = f(x)$ is the function defined by

$$y' = y^2 + xy, \quad y(0) = 1.$$ 

Obtain a MacLaurin polynomial for $f$.

$$y(0) = 1$$
$$y'(0) = 1$$
$$y''(0) = 3$$
$$y'''(0) = 10$$

$$y(x) = 1 + x + \frac{3}{2!} x^2 + \frac{10}{3!} x^3 + \ldots$$
Equation Solving

- Think of $y$ as a function of $x$ being defined implicitly by the equation
  \[ x^2 + y^2 = 1 \]
  \[ y = \pm \sqrt{1 - x^2} \]

Compute a MacLaurin Polynomial and compare it with the true solution.

\[ 2x + 2y \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{x}{y} \]

\[ 2 + 2 \left( y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = 0 \]

\[ 2 \frac{d^2y}{dx^2} = -2 - 2 \left(\frac{dy}{dx}\right)^2 \]

\[ \frac{d^2y}{dx^2} = -\frac{1 + y^2}{y} \]

\[ -1 = y''(0) \]

\[ y''' = -\frac{y - xy'}{y^2} \]

\[ -1 = y'''(0) \]

\[ y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \ldots \]

\[ = 1 - \frac{x^2}{2} \]
Figure 1. MacLaurin Polynomials of $y = \sqrt{1 - x^2} \approx 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}$.

Of course, in this case we know the solution of the equation and we could have computed the MacLaurin polynomial directly by computing derivatives of

$$f(x) = \sqrt{1 - x^2}.$$ 

But the technique illustrated here works even if we don’t know the solution and therefore can’t plot it.
• Let’s look more closely at the quality of the polynomial approximation.

\[ f(x) = T_n(x) + R_n(x) \]

where the \( n \)-degree Taylor polynomial for \( f \) about \( a \) is

\[ T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k \]

\[ = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n \]

and

\[ R_n(x) = f(x) - T_n(x) \]

is the remainder.

• The remainder can be written in several different ways.

• The integral form is

\[ R_n(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t) \, dt \]

and

• Lagrange’s form says that there exists a number \( z \) between \( a \) and \( x \) such that

\[ R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - a)^{n+1} \]

Thus the remainder looks exactly like the next term in the Taylor series, except that the derivative is evaluated at an unknown point which we know to exist.

• Immediate consequences: If the derivatives are bounded the Remainder converges to zero. For example, the Taylor series for the exponential, \( \sin \), and \( \cos \) all have an infinite radius of convergence.
Derivation of Taylor’s formula

- Recall integration by parts:

\[ \int_a^b u'(t)v(t)dt = u(t)v(t) \bigg|_a^b - \int_a^b u(t)v'(t)dt. \]

- We get

\[
\begin{align*}
f(x) &= f(0) + \int_0^x f'(t)dt \\
&= f(0) + \int_0^1 1 \times f'(t)dt \\
&\quad \text{integration by parts, } u'(t) = 1, \ u(t) = t - x, \ v(t) = f'(t) \\
&= f(0) + (t - x)f'(t) \bigg|_0^x - \int_0^x (t - x)f''(t)dt \\
&\quad \text{integration by parts again} \ldots \\
&= f(0) + xf'(0) - \left[ \frac{1}{2}(t - x)^2 f''(t) \right]_0^x + \int_0^x \frac{1}{2} (t - x)^2 f'''(t)dt \\
&\quad \text{and again} \ldots \\
&= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \left[ \frac{1}{6}(t - x)^3 f'''(t) \right]_0^x - \int_0^x \frac{1}{6} (t - x)^3 f''''(t)dt \\
&\quad \ldots \text{leading to:} \\
&= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + R_n(x)
\end{align*}
\]

where

\[
R_n(x) = \frac{(-1)^n}{n!} \int_0^x (t-x)^n f^{(n+1)}(t)dt = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)dt.
\]
\[ f(x) = 7 + 3x + 4x^2 + 4x^3 + \ldots \]
\[ g(x) = 35 + 36x + 64x^2 + 75x^3 + \ldots \]
\[ \frac{g(x)}{f(x)} = a + bx + cx + d \]

\[(a + bx + cx + d)(4x^3 + 4x^2 + 3x + 2)\]

\[7a + 7b + 7c + 7d + 3ax^4 + 4bx^3 + 3cx^2 + 3dx + 3dx^2 + 4dx^2\]

\[= (7b + 3c + 4d)x + (7c + d) + 7d\]

\[7d = 35 \quad d = 5\]
\[7c + 3d = 36 \quad 7c + 15 = 36 \quad c = 3\]