8.6 Power Series as Functions

- The basic fact is that the power series

\[ s(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \]

can be treated like a polynomial on its interval of convergence.

- Today: \( a = 0 \) throughout.

- Example:

\[ s(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]
Figure 1. Graphs of Exponential and $p_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$, $n = 1, 2, 3, 4, 5$.

Figure 2. Graphs of Exponential and $p_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$, $n = 1, 2, 3, 4, 5$. 
• Recall
\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \quad |x| < 1. \]

• Example 5: Compute the power series of
\[ f(x) = \frac{1}{(1-x)^2} \]
\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \quad |x| < 1.
\]

- Example 6: Compute the power series of

\[ f(x) = \ln(x + 1) \]
\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \quad |x| < 1. \]

- Example 1: Find the power series for

\[ f(x) = \frac{1}{1 + x^2} \]
\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \quad |x| < 1. \]

- Example 7: Find the power series for

\[ f(x) = \arctan x \]
8.7 Taylor Series

• We need a more systematic approach. Key idea: match derivatives.

• Derivatives of $x^n$ at $x = 0$. 
• Suppose we want

\[ s(x) = \sum_{n=0}^{\infty} c_n x^n \]

have the same derivatives as a function \( f \) at \( x = 0 \):

\[ s^{(n)}(0) = f^{(n)}(0), \quad n = 0, 1, 2, \ldots \]

• How do we pick the \( c_n \)?