Reminders

1.3 New Functions from Old Functions

- Translations (particularly horizontal and vertical shifts)
- Reflections
- Scalings
- Arithmetic Operations
- Composition and Decomposition
- Example:
  for Translations: Graphs of

\[ f(x) = x^2, \quad g(x) = x^2 + 2, \quad h(x) = (x + 2)^2. \]

\[ = f(x + 2) \]

\[ = f(x) + 2 \]

\[ y = f(x) \]

\[ y = f(x) + 2 \leq \]

\[ y = f(x + 2) \]

\[ y = 2 \leq \]

\[ y = f(x - 2) \]

**Figure 1.** Graphs of \( x^2, x^2 + 2, (x + 2)^2 \).
Figure 2. Graphs of $x^2$, $x^2 + 2$, $(x + 2)^2$. 
Reflections

- Example:

\[ f(x) = e^x, \quad g(x) = f(-x) = e^{-x}, \quad \text{and} \quad h(x) = -f(-x) = -e^{-x}. \]

Figure 3. Graphs of \( e^x, e^{-x}, \) and \( -e^{-x}. \)
• Example:

\[ f(x) = e^x, \quad g(x) = f(-x) = e^{-x}, \quad \text{and} \quad h(x) = -f(-x) = e^{-x}. \]

Figure 4. Graphs of \( e^x, e^{-x}, \) and \( -e^{-x}. \)
Symmetry

- Functions whose graphs are symmetric with respect to the \( y \) axis are called **even functions**.
- Functions whose graphs are symmetric with respect to the origin are called **odd functions**.

**Examples:**

\[
\begin{align*}
\text{even} & \quad f(x) = x^2 \quad f(x) = 1 \quad f(x) = x^3 \quad f(x) = 1 - x \\
\cos x & \\
\text{odd} & \quad f(x) = x, \quad x, \quad x, \quad x \\
\sin x & \quad \frac{1}{x} = x \quad f(x) = \frac{1}{x} \quad f(-x) = \frac{1}{x} \quad \frac{1}{x} \quad -f(x)
\end{align*}
\]

\[
\sin(-x) = -\sin x
\]

\[
\text{Graph of function}
\]
• Query: We discussed functions whose graphs are unchanged by reflections in the $y$-axis or the origin. Why does the textbook (or anybody else) not mention, let alone discuss, functions whose graph remains the same if reflected in the $x$-axis?

\[ f(x) = 0 \]

\[ f(x) = f(-x) = -f(-x) \]

\[ f(x) = 0 \]
Scalings

- You can think of this as changing the units on the horizontal or vertical axis.
- \( \hat{f}(x) = f(cx) \) amounts to a horizontal rescaling.
- \( \hat{f}(x) = cf(x) \) amounts to a vertical rescaling.
- Sometimes vertical and horizontal scalings are equivalent. For example in \( y = g(x) = 2x^2 = (\sqrt{2}x) \) we can think of the graph as obtained by scaling the graph of \( f(x) = x^2 \) vertically by 2, or horizontally by \( 1/\sqrt{2} \).

- Exercise: review and consider examples if necessary.
Arithmetic Combinations

\[(f \triangle g)(x) = f(x) \triangle g(x) \quad \text{where} \quad \triangle = +, -, \times, /.

- **Example:**
  
  \[f(x) = x + 1 \quad \text{and} \quad g(x) = x^2\]
  
  • \((f + g)(x) = x + 1 + x^2\)
  
  • \((f - g)(x) = x + 1 - x^2\)
  
  • \((f \times g)(x) = (x + 1)x^2 = x^3 + x^2\)
  
  • \((f / g)(x) = \frac{x + 1}{x^2}\)
Composition of Functions

\[(f \circ g)(x) = f(g(x))\]

- **Example:**
  
  \[f(x) = x + 1 \text{ and } g(x) = x^2\]

  - \[(f \circ f)(x) = x + 1 + 1 = x + 2\]
  - \[(f \circ g)(x) = f(g(x)) = f(x^2) = x + 1\]
  - \[(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^2\]
  - \[(g \circ g)(x) = (x + 1)^2\]

\[(x + 1)^2 = (x + 1)(x + 1) = x^2 + 2x + 1 \neq (x + 1)^2\]

**Note!**

Function composition does not commute
• **Example:**

\[ f(x) = \sin x \text{ and } g(x) = x^2 \]

• \((f \circ g)(x) = \sin \left(x^2\right) = \sin(x^2)\)

• \((g \circ f)(x) = \left(\sin x\right)^2 = \sin^2 x\)
• Example:
\[ f(x) = x + 1, \ g(x) = x^2, \] and \[ h(x) = \frac{1}{x}. \]

\[ (f \circ g \circ h)(x) = \left( f \circ g \right) \left( \frac{1}{x} \right) = f \left( g \left( \frac{1}{x} \right) \right) = f \left( \frac{1}{x^2} \right) = \frac{1}{x^2} + 1 \]

\[ (h \circ g \circ f)(x) = \left( h \circ g \right) \left( x + 1 \right) = h \left( g \left( x + 1 \right) \right) = \frac{1}{(x + 1)^2} \]

• Exercise: compute \( f \circ h \circ g, \ g \circ f \circ h, \ g \circ h \circ f \) and \( h \circ f \circ g. \)
Decomposition of Functions

- Example:

\[(x^2 + 1)^{\frac{1}{2}} = (f \circ g)(x)\]

\[= f(g(x))\]

\[f(x) = x^{\frac{1}{2}}\]

\[g(x) = x^2\]

\[\text{Solution:} \ 1\]

\[f(x) = (x+1)^{\frac{1}{2}}\]

\[g(x) = x^2\]

\[h(x) = x + 1\]

\[f(x) = x^{\frac{1}{2}}\]

\[g(x) = x + 1\]

\[h(x) = x^2\]
Summary

- Suppose $f$ and $g$ are given functions.
- $\hat{f}(x) = f(x - c)$: Horizontal shift of the graph of $f$ by $c$. (horizontal shift of origin by $-c$).
- $\hat{f}(x) = f(x) + c$: Vertical shift of the graph of $f$ by $c$. (vertical shift of origin by $-c$).
- $\hat{f}(x) = f(-x)$: Reflection in (or about) the $y$-axis. The function is **even** if $f(x) = f(-x)$.
- $\hat{f}(x) = -f(x)$: Reflection in (or about) the $x$-axis.
- $\hat{f}(x) = -f(-x)$: Reflection in (or through) the origin. The function is **odd** if $f(x) = -f(-x)$.
- $\hat{f}(x) = f(cx)$ horizontal rescaling (change of units on horizontal axis).
- $\hat{f}(x) = cf(x)$ vertical rescaling (change of units on vertical axis).
- Arithmetic combination of functions:

$$f(f \triangle g)(x) = f(x) \triangle g(x) \quad \text{where} \quad \triangle = +, -, \times, /.$$

- Composition of functions:

$$(f \circ g)(x) = f(g(x))$$