Some Rope

- This may be my favorite Calculus problem. It certainly is a spectacular application of Calculus!

- It’s also a cool Engineering problem!

- Suppose you want to build a rope that can reach a very great depth.

- The trouble with an ordinary, cylindrical, rope is that if it is too long it will break under its own weight.

- The length at which it would break is independent of its radius, since as the radius increases so does the weight of the rope. The two processes exactly balance.

- Let’s look a little more closely. The rope will break when the strain at some point exceeds a critical value that depends on the material of which the rope is made. Thus the rope will break when

\[
\text{strain} = \frac{\text{weight}}{\text{area of cross-section}} > \text{critical strain}
\]

- To give a physical flavor to these calculations consider stainless steel with a specific weight of 500 pounds per cubic foot and a critical strain of 720,000 pounds per square foot\(^{-1}\).

- Let’s also suppose we want our rope to be able to lift a weight of 1000 pounds attached at the bottom of the rope.

- Now consider an ordinary cylindrical rope of length \(L\) and radius \(r\). The weight of that rope equals

\[
w = \pi r^2 L \delta
\]

\(^{-1}\) I googled those numbers and don’t know how reasonable or accurate they are.
where \( \delta \) is the specific weight of that rope. The area of the cross section is

\[
A = \pi r^2
\]

and so we obtain the condition

\[
\frac{\pi r^2 L \delta}{\pi r^2} = L \delta > \gamma
\]

or

\[
L > \frac{\gamma}{\delta} = L_0
\]

where \( \gamma \) is the critical strain and the critical length \( L_0 \) is independent of the radius of the rope.

- In our example,

\[
\gamma = 720,000 \text{ pounds/square foot},
\]
\[
\delta = 500 \text{ pounds/cubic foot},
\]
\[
L_0 = 14,400 \text{ feet}.
\]

- So our rope would break under its own weight if it was longer than 14,400 feet, or about 3 miles.

- However, we can increase the depth a rope can reach beyond the critical length by increasing the radius of the rope towards the top!

- So suppose we let the radius of the rope be \( r(x) \) where \( x \) is the length of the rope measured upwards from the bottom.

- The weight of the rope at a point \( x \) equals the volume of the rope below \( x \) multiplied with the specific weight \( \delta \).

- We want to find the function \( r(x) \) that makes the strain \( \gamma(x) \) in the rope constant, i.e., \( \gamma(x) = \gamma \).

- If we were designing an actual rope we would want that constant strain to be well below the critical value, but for our numerical example let’s stick with \( \gamma = 720,000 \) pounds per square inch.
Let’s also suppose that at the bottom of the rope it will carry a weight \( w_0 = 1000 \) pounds, say.

The volume of the rope can be computed easily by considering it a solid of revolution.

Putting this information into mathematical terms we obtain the equation

\[
\frac{\delta}{\pi} \int_0^x \pi r^2(t) dt + w_0 = \frac{\gamma}{\pi r^2(x)}
\]

or

\[
\frac{d}{dx} \left( \frac{\delta}{\pi} \int_0^x \pi r^2(t) dt + w_0 \right) = \gamma \pi r^2(x)
\]

(1)

Differentiating in (1) gives (by the fundamental theorem of Calculus)

\[
\delta \pi r^2(x) = 2\pi \gamma r(x) r'(x).
\]

Solving for \( r'(x) \) gives the differential equation

\[
r'(x) = \frac{\delta}{2\gamma} r(x)
\]

which has the solution

\[
r(x) = C \exp\left( \frac{\delta}{2\gamma} x \right)
\]

(\( \exp(x) \) denotes the exponential, \( \exp(x) = e^x \)).

To determine \( C = r(0) \) we set \( x = 0 \) in (1) and obtain

\[
w_0 = \gamma \pi r^2(0)
\]

which gives

\[
r(0) = \sqrt{\frac{w_0}{\gamma \pi}} \approx \zeta
\]
Hence

\[ r(x) = \sqrt{\frac{w_0}{\gamma \pi}} \exp \left( \frac{\delta}{2 \gamma} x \right) \]

which is the formula that we have been looking for!

In our example,

\[ \gamma = 720,000, \quad \delta = 500, \quad \text{and} \quad w_0 = 1000. \]

Thus

\[ r(x) = \sqrt{\frac{1000}{\pi \times 720,000}} \exp \left( \frac{500}{1,440,000} x \right) \]

The following table gives the diameter of our rope for some values of \( x \):

<table>
<thead>
<tr>
<th>( x ) (miles)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x ) (feet)</td>
<td>0</td>
<td>5,280</td>
<td>10,560</td>
<td>15,840</td>
<td>21,200</td>
<td>26,400</td>
</tr>
<tr>
<td>( r(x) ) (feet)</td>
<td>0.021</td>
<td>0.132</td>
<td>0.823</td>
<td>5.144</td>
<td>32.180</td>
<td>201.275</td>
</tr>
<tr>
<td>( r(x) ) (inches)</td>
<td>0.252</td>
<td>1.578</td>
<td>9.871</td>
<td>61.739</td>
<td>386.159</td>
<td>2415.310</td>
</tr>
<tr>
<td>weight of rope (pounds)</td>
<td>0</td>
<td>3.8E4</td>
<td>1.53E6</td>
<td>5.99E7</td>
<td>2.34E9</td>
<td>9.6E10</td>
</tr>
</tbody>
</table>

Some Rope!

If that rope was 5 miles long it would weigh about \( 10^{11} \) pounds, or 50 million tons. For the fun of it, I googled the annual world steel production. In the fall of 2019 this was listed on the wikipedia as 1808.6 million tons. At that rate it would take more than \textbf{10 days} for the world to produce enough steel to build that rope!