Reminder

- The “MathCenter” (formally the “T. Benny Rushing Mathematics Students Center”) offers free tutoring, pleasant study space, and computer lab for your use.

- It is located underneath the plaza between LCB and JWB on President’s Circle.

- For more info, including a list of tutors and their schedules, see

  http://www.math.utah.edu/undergrad/mathcenter.php

Undergraduate Colloquium
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We 12:55 LCB 225

Report on an Arctic Expedition
5.6 Integration by Parts

• Integration by parts is the reverse process of differentiation by the product rule.

\[
\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).
\]

• For example, the differentiation formula

\[
\frac{d}{dx} e^x \sin x = e^x \sin x + e^x \cos x
\]
gives the integration formula

\[
\int e^x \sin x + e^x \cos x \, dx = e^x \sin x + C.
\] (1)

However, this straight inversion of the product rule is not useful. Instead of

\[
f(x)g(x) = \int f'(x)g(x) + f(x)g'(x) \, dx
= \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx
\]

we use the equivalent form

\[
\int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx
\]

• Of course, the roles of \(f\) and \(g\) may be interchanged:
\[ \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \]

- These formulas can be abbreviated, for example as

\[ \int fg' = fg - \int f'g \]

- Frequently used instead of \( f \) and \( g \) are \( u \) and \( v \):

\[ \int uv' = uv - \int u'v \] \hspace{1cm} (2)

Also used, but, I think, exceedingly confusing, is formula 2 on page 383 of the textbook. Letting \( u = f(x) \), \( du = f'(x)dx \), \( v = g(x) \), and \( dv = g'(x)dx \) we get

\[ \int udv = uv - \int vdu. \] \hspace{1cm} (3)

This looks like we are integrating \( u \) with respect to \( v \) and \( v \) with respect to \( u \) which makes no sense at all!

- The notation in formula (3) is as bad as using a superscript -1 to denote the inverse function. Luckily we have alternatives for the integration formula.

- I would avoid this formula because it suggests we are integrating \( u \) with respect to \( v \) which does not make sense.
• You can always go from (3) to (2) by “multiplying and dividing” with $dx$ under the integrals in (3):

$$\int u dv = \int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v dx = uv - \int v du.$$  

• Applying the formula (2) or one of its equivalent forms is called **integration by parts**.

It’s hard to appreciate at this stage of our investigation, but Integration by parts may be considered the most important integration technique there is.

• There is also a variant of integration by parts for definite integrals:

$$\int_a^b f' g = fg|_a^b - \int_a^b fg'$$

Note the Pattern: We integrate a product, and we need to compute the derivative of one factor and an antiderivative of the other factor.

We pick which is which so as to simplify the problem!

• Examples ....
\[ I = \int x e^x \, dx \]

\[
\begin{align*}
uv' & = uv - \int v u' \, dx \\
& = x e^x - e^x + C
\end{align*}
\]

\[ \int = e^x + x e^x - e^x = x e^x \checkmark \]
\[ I = \int e^x \sin x \, dx = \int e^x \sin x \, dx - \int e^x \cos x \, dx \]

\[ = e^x \sin x - \left[ e^x \cos x + e^x \sin x \, dx \right] \]

\[ I = e^x \sin x - e^x \cos x - I \]

\[ 2I = e^x (\sin x - \cos x) \]

\[ I = \frac{1}{2} e^x (\sin x - \cos x) + C \]

\[ I' = \frac{1}{2} \left( e^x (\sin x - \cos x) + e^x (\cos x + \sin x) \right) \]

\[ = e^x \sin x \]
\[ I = \int \ln x \, dx = \int e^{\ln x} \, dx \]

\[ = \int u' \, u \, dx \]

\[ = x \ln x - \int x \frac{1}{x} \, dx \]

\[ = x \ln x - \int 1 \, dx \]

\[ = x \ln x - x + C \]

\[ \text{CBD} \]

\[ = \ln x + x \cdot \frac{1}{x} - 1 \]

\[ = \ln x \]
\[ I = \int \arctan x \, dx \]

\[
= \int u \, du
\]

\[
= x \arctan x - \int \frac{x}{1 + x^2} \, dx
\]

\[
= x \arctan x - \frac{1}{2} \ln (1 + x^2)
\]

\[
\bar{I} = \arctan x + \frac{x}{1 + x^2} - \frac{1}{2} \frac{2x}{1 + x^2}
\]

\[
= \arctan x
\]

Ex.: \[ \int \frac{1}{\sqrt{1 - x^2}} \, dx \]
Added after class. Integration of \( \frac{x}{1+x^2} \) by substitution.

\[
I = \int \frac{x}{1+x^2} \, dx \\
= \frac{1}{2} \int \frac{1}{u} \, du \\
= \frac{1}{2} \ln |u| + C \\
= \frac{1}{2} \ln (1+x^2)
\]

Technically, we have to worry about \( u \) being negative but of course \( 1+x^2 \) is always positive.
\[ I_n = \int_{-\pi}^{\pi} \sin^n x \, dx \]

\[
\begin{align*}
I_0 &= \int_{-\pi}^{\pi} 1 \, dx = [x]_{-\pi}^{\pi} = 2\pi \\
I_1 &= \int_{-\pi}^{\pi} \sin x \, dx \\
&= n \text{ odd} \quad \sin^n x \text{ odd} \\
&= 0 \quad \text{if } n \text{ is odd} \\
I_n &= \int_{-\pi}^{\pi} \sin^n x \, dx \\
&= \int_{-\pi}^{\pi} \sin x \, \sin^{n-1} x \, dx \\
&= \int_{-\pi}^{\pi} u^{\prime} \, u \, du \\
&= -\cos x \sin^{n-1} x \bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos^2 (n-1) \sin^{n-2} x \, dx \\
&= \int_{-\pi}^{\pi} \cos^n x \, dx \\
&= \int_{-\pi}^{\pi} \cos^{2n} x \, dx
\]
\[ = (n-1) \int_{-\pi}^{\pi} \cos^2 x \sin^{n-2} x \, dx \]

\[ \int_{-\pi}^{\pi} = (n-1) \left( \int_{-\pi}^{\pi} (1 - \sin^2 x) \sin^{n-2} x \, dx \right) \]

\[ \int_{-\pi}^{\pi} = (n-1) \left( \int_{-\pi}^{\pi} \sin^{n-2} x \, dx \right) \]

\[ = (n-1) \int_{-\pi}^{\pi} \sin^{n-2} x \, dx \]

\[ = (n-1) I_{n-2} - (n-1) I_n \]

\[ I_n = (n-1) I_{n-2} - n I_n + I_n \]

\[ n I_n = (n-1) I_{n-2} \]

\[ I_n = \frac{n-1}{n} I_{n-2} \]

\[ I_0 = 2\pi I_2 = \frac{1}{2} I_0 = \pi \]

\[ I_4 = \frac{3}{2} \pi I_6 = \frac{5}{6} \cdot \frac{3}{2} \pi = \frac{5}{8} \pi \ldots \]