• Recall our

**Definition:** If \( f \) is a function defined for \( a \leq x \leq b \), we divide the interval \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = (b - a)/n \). We let \( a = x_0 < x_1 < \ldots < x_n = b \) where \( x_i = a + i\Delta x \) be the endpoints of these intervals and we let \( x_1^*, x_2^*, \ldots, x_n^* \) be any **sample points** in these subintervals, so \( x_i^* \) lies in the \( i \)-th subinterval \([x_{i-1}, x_i]\). Then the **definite integral of \( f \) from \( a \) to \( b \)** is

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

provided that this limit exists. If it does exist we say that \( f \) is **integrable** on \([a, b]\).

• Recall our terminology:
  - \( \int \) is the **integration symbol** or **integral sign**.
  - \( x \) is the **integration variable**.
  - \( f \) is the **integrand**.
  - \( a \) and \( b \) are the **lower** and **upper limit of integration**, respectively.

• Our geometric motivation for defining the integral was to compute the area underneath the graph of a (positive) function \( f \).
5.3 Evaluating Definite Integrals

- **Evaluation Theorem** (p. 356, textbook) If \( f \) is continuous on the interval \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

where \( F \) is any antiderivative of \( f \), i.e., \( F' = f \).

- This is plausible if \( x \) is time, \( f(x) \) is velocity, and \( F(x) \) is location.

- We can also see it using the mean value theorem.

- if \( F \) is differentiable on \([a, b]\) then there exists a \( c \) in \((a, b)\) such that

\[
F(b) - F(a) = F'(c)(b - a).
\]

- With a slight change in notation:

\[
F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}).
\]

- We now get, with \( \Delta x = \frac{b-a}{n} \) and \( x_i = a + i\Delta x \)

\[
F(b) - F(a) = F(x_n) - F(x_0) = (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \ldots
\]

\[
\ldots + (F(x_2) - F(x_1)) + (F(x_1) - F(x_0))
\]

\[
= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))
\]

\[
= \sum_{i=1}^{n} (x_i - x_{i-1}) F'(x_i^*)
\]

\[
= \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

The limit of that sum is \( \int_a^b f(x) \, dx \).
More Notation:

\[ \int_a^b f(x) dx = F(b) - F(a) = \left[ F(x) \right]_a^b = F(x) \bigg|_a^b \]

Examples:

\( \bullet \) \( \int_0^1 x^n dx = \left[ \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1} \quad n \neq -1 \)

\( \bullet \) \( \int_0^\pi \sin x dx = \left[ -\cos x \right]_0^\pi = -\cos \pi + \cos 0 = 1 + 1 = 2 \)

\( \bullet \) \( \int_{-1}^1 (1 - x^2) dx = \left[ x - \frac{x^3}{3} \right]_{-1}^1 = 1 - \frac{1}{3} - \left( -1 - \frac{(-1)^3}{3} \right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)

\( \bullet \) \( \int_{-1}^1 \sqrt{1 - x^2} dx = \frac{\pi}{2} \)

\[ \frac{1}{2} \left[ x \sqrt{1-x^2} + \arcsin x \right]_{-1}^1 = \frac{1}{2} \left( \arcsin 1 - \arcsin (-1) \right) = \frac{1}{2} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \]
The Evaluation Theorem gives the same answer for the last problem, except that we don’t yet know how to compute an antiderivative. However, one can check by differentiation that an antiderivative of $f(x) = \sqrt{1 - x^2}$ is

$$F(x) = \frac{1}{2} \left[ x\sqrt{1 - x^2} + \arcsin x \right]$$

$$F'(x) = \frac{1}{2} \left[ \sqrt{1 - x^2} \cdot \frac{2x^2}{2\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}} \right]$$

$$= \frac{1}{2} \left[ \frac{1 - x^2 - x^2 + 1}{\sqrt{1 - x^2}} \right]$$

$$= \frac{1}{2} \left[ \frac{2(1 - x^2)}{\sqrt{1 - x^2}} \right] = \frac{1 - x^2}{\sqrt{1 - x^2}} = \sqrt{1 - x^2}$$
Indefinite Integrals

page 358, textbook:

\[ \int f(x) \, dx = F(x) \quad \text{means} \quad F'(x) = f(x). \]

- \( \int f(x) \, dx \) is called an **indefinite integral**. Depending on the context it stands for a particular antiderivative or for all antiderivatives.

- An indefinite integral is a function or a family of functions, a definite integral is a number.

- A definite integral has limits of integration, an indefinite does not.

- Often (like in the back of our textbook) you will find the notation
  \[ \int f(x) \, dx = F(x) + C \]
  as in
  \[ \int x^2 \, dx = \frac{x^3}{3} + C \]
  where \( C \) is meant to stand for an arbitrary constant, called the **integration constant**.

- For our purposes, we need to be always aware that an antiderivative is determined only up to a constant.

- Clearly, we need a supply of antidifferentiation (integration) rules.
Important point: Every differentiation rule is also an integration rule.

- Following is a restatement of (most of) our differentiation rules as integration rules:

\[
\begin{align*}
\int [f(x) + g(x)]\,dx &= \int f(x)\,dx + \int g(x)\,dx \\
\int cf(x)\,dx &= c \int f(x)\,dx \\
\int x^n\,dx &= \frac{x^{n+1}}{n+1} + C \\
\int e^x\,dx &= e^x + C \\
\int \frac{1}{x}\,dx &= \ln x + C \\
\int \sin x\,dx &= -\cos x + C \\
\int \cos x\,dx &= \sin x + C \\
\int \frac{1}{1 + x^2}\,dx &= \arctan x + C \\
\int \frac{1}{\sqrt{1 - x^2}}\,dx &= \arcsin x + C \\
\int f'(g(x))g'(x)\,dx &= f(g(x)) + C \\
\int f'(x)g(x) + f(x)g'(x)\,dx &= f(x)g(x) + C \\
\int \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}\,dx &= \frac{f(x)}{g(x)} + C
\end{align*}
\]

Also note that

\[
\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.
\]
Displacement versus distance traveled

Example 7, page 362 textbook.

A particle moves along a line so that its velocity at time $t$ is

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$$

(measured in meters per second).

(a) Find the displacement of the particle during the time interval $1 \leq t \leq 4$

(b) Find the distance traveled by the particle during this time period.

• We first note that

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$$

In the interval of interest, $[1,4]$, the particle changes direction, from negative to positive, at $t = 3$.

• To compute the displacement we simply take the difference between the locations $s(4)$ and $s(1)$, where $s' = v$:

$$s(4) - s(1) = \int_{1}^{4} v(t)dt$$

$$= \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_{1}^{4}$$

$$= \left( \frac{4^3}{3} - \frac{4^2}{2} - 6 \times 4 \right) - \left( \frac{1^3}{3} - \frac{1^2}{2} - 6 \times 1 \right)$$

$$= \frac{128}{3} - \frac{16}{2} - 24 - \left( \frac{1}{3} - \frac{1}{2} - 6 \right)$$

$$= \frac{128 - 48 - 144 - 2 + 3 + 36}{6}$$

$$= \frac{-27}{6}$$

$$= -\frac{9}{2}$$

• The particle ends up 4.5 meters to the left of where it started.
• To compute the total traveled distance $D$ (what the particle’s odometer would show) we add the distance covered in the interval $[1, 3]$ and the distance traveled in the interval $[3, 4]$. Noting that initially we travel to the left we get

$$D = -\int_1^3 v(t)dt + \int_3^4 v(t)dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \ldots$$

$$= \frac{61}{6}$$

• The total distance traveled is a little more than 10m, more than twice the actual displacement.
What about \( \frac{d}{dx} \int_a^x f(t)dt \)?

\[
\frac{d}{dx} \int_a^x f(t)dt = \frac{d}{dx} \left[ F(x) - F(a) \right]
\]

\[
= f(x)
\]

\[
\frac{d}{dx} \text{ constant } = 0
\]

\[
\frac{d}{dx} \int_a^x f(t)dt = f(x)
\]

\[
f(t) = t
\]

\[
\int_a^x t \, dt = \left[ \frac{t^2}{2} \right]_a^x = \frac{x^2}{2} - \frac{a^2}{2}
\]

\[
\frac{d}{dx} \left[ \frac{x^2}{2} - \frac{a^2}{2} \right] = x
\]
\[
\frac{d}{dx} \int_1^x t \, dt = x^2
\]

\[
\frac{d}{dx} \int_1^{x^2} t \, dt = \frac{d}{dx} \left[ \frac{t^2}{2} \right]_1^{x^2} = \frac{d}{dx} \left[ \frac{x^4}{2} \right] = \frac{4x^3}{2} - 2x^3
\]

\[
\frac{d}{dx} \int_1^{f(x)} dx = \frac{d}{dx} \left( F(x^2) - F(1) \right)
\]

\[
= \frac{d}{dx} F(x^2) \cdot 2x
\]

\[
x^2 \cdot 2x = 2x^3
\]

\[
\frac{d}{dx} \int_1^{x^2} t \, dt = (x^2)^2 \cdot 2x
\]
\# 12

\[ x = 4.3 \cos \kappa \]

\[ f(x) = 0 \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ f(x) = x - 4.3 \cos \kappa \]

\[ x_0 = 1 \quad x_{n+1} = x_n - \frac{1 - 4.3 \cos \kappa}{1 + 4.3 \sin \kappa} \]