4.8 Antiderivatives

- We know how to go from a function to its derivative.
- We can also go the other way!
- **Definition**, page 317. A function \( F \) is an **antiderivative** of a function \( f \) if \( F'(x) = f(x) \).
- **Examples**: Find \( F \) given \( f \):

  \[
  \begin{align*}
  f(x) &= 1 & F(x) &= x + 1 + C \\
  f(x) &= x + 1 & F(x) &= \frac{1}{2}x^2 + x + C \\
  f(x) &= \cos x & F(x) &= \sin x + C \\
  f(x) &= e^x & F(x) &= e^x + C
  \end{align*}
  \]

- Antiderivatives are determined only up to a constant.
- **Theorem**, page 317. If \( F \) is an antiderivative of \( f \) on an interval \( I \), then the most general antiderivative of \( f \) on \( I \) is

  \[ F(x) + C \]

  where \( C \) is an arbitrary constant.
A function has infinitely many antiderivatives. They differ by constants. Geometrically, their graphs are vertical translates of each other. This is illustrated in Figure 1 which shows the function

\[ f(x) = x \]

and some of its antiderivatives

\[ F(x) = \frac{x^2}{2} + C. \]

Note that all antiderivatives have the same slope for a given value of \( x \).
• While a function has infinitely many Antiderivatives we may be interested in a particular one that is determined by an additional condition.

• **Examples:**

Again, give $F' = f$, find $F$

$f(x) = x$ and $F(1) = 2$

$$F(x) = \frac{x^2}{2} + C$$

$$F(1) = 2 = \frac{1}{2} + C \quad \Rightarrow \quad C = \frac{3}{2}$$

$f(x) = \sin x$ and $F\left(\frac{\pi}{2}\right) = 5$

$$F(x) = -\cos x + C$$

$$F\left(\frac{\pi}{2}\right) = -\cos \frac{\pi}{2} + C = C = 5$$

$f(x) = m$ and $F(0) = b$

$$F(x) = mx + C$$

$$F(0) = m \cdot 0 + C = C = b$$

$$F(x) = mx + b$$
A physical example. Ignoring air resistance, falling objects on earth accelerate at 32 feet per second squared. Derive expressions for their velocity \( v(t) \) and height \( h(t) \).

\[
\begin{align*}
a &= -32 \\
v(t) &= -32t + v_0 \\
v(t) &= -32t + v_0 \\
h(t) &= -16t^2 + v_0 t + h_0 \\
h(t) &= -16t^2 + v_0 t + h_0
\end{align*}
\]
Cruising Down the Highway

- Problem 53, page 322, modified for a Utah freeway. A car is traveling at 80mph when the driver sees an accident 300 feet ahead and slams on the brakes. What constant deceleration is required to avoid a pile up? What about traveling at 100mph? A mile is 5280 feet and an hour is 3600 seconds.

- Let’s do this problem in general and substitute numbers at the end.

- Suppose the distance from the accident is \( s(t) \), the velocity is \( v(t) \), and the acceleration is \( a(t) = a \) where \( a \) is the constant to be determined. Let’s denote the initial velocity by \( v_0 \) and the initial distance (the distance at which you start braking at time \( t = 0 \)) by \( s_0 \).

- We get

  \[
  a(t) = a \\
  v(t) = at + v_0 \\
  s(t) = a\frac{t^2}{2} + v_0 t + s_0
  \]

- We want to compute \( a \) for which \( v(t) \) is zero exactly when \( s(t) \) is zero. Let’s denote by \( t_0 \) the time at which this happens. As usual in this kind of problems we first compute \( t_0 \) and then compute \( a \).

- Setting

  \[
  v(t) = at + v_0 = 0
  \]

  and solving for \( t \) gives

  \[
  t = t_0 = -\frac{v_0}{a}.
  \]

- This is actually a positive quantity since the initial velocity (rate at which the distance from the accident is changing) is negative. \( t_0 \) is the time at which the car comes to a stop.
stop. We evaluate the distance from the accident at that
time, set it equal to 0 and solve for \(a\). This gives

\[
s(t_0) = a \frac{t_0^2}{2} + v_0 t_0 + s_0
\]

\[
= a \frac{v_0^2}{2a^2} - v_0 \frac{v_0}{a} + s_0 = \frac{1}{2} \frac{v_0^2}{a} - \frac{v_0^2}{a} + s_0
\]

\[
= -\frac{v_0^2}{2a} + s_0
\]

\[
= 0
\]

\[
\frac{1}{2} \frac{v_0^2}{a} = s_0
\]

\[
\frac{a}{v_0^2} = \frac{1}{2} \frac{v_0^2}{2s_0}
\]

\[
a = \frac{v_0^2}{2s_0}
\]

- Thus

- Converting the velocities from mph to feet per second gives
  the following table of numerical values for various speeds
  and distances.

- Each entry gives two values, the time \(t_0\) giving the time
  in seconds until reaching the accident, and the necessary
deceleration, measured in multiples of \(g = 32\) feet per
  second squared.
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<th>dist (feet)</th>
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<th>200</th>
<th>250</th>
<th>300</th>
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<td>1.34</td>
<td>1.12</td>
</tr>
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</table>

Table: Braking Times (seconds, top) and decelerations (g, bottom)

• According to a Google query, a typical stopping distance from 60mph might be about 150 feet. This corresponds to a deceleration of 0.81g, so you can’t expect to be able to brake harder than that! And of course the stopping distance increases greatly when there is snow on the road!

Note that this table does not take into account your reaction time, i.e., the time that will pass between your noticing the accident and starting to brake.
Disclaimer: These calculations are meant as a mathematical exercise, not as advice, promise, or guarantee, for your actual driving. Your stopping distance will vary and depends on your car, the manner of your braking, the condition of the road, and your reaction time. When operating a vehicle, always stay alert and focused, keep a safe distance, and travel at a safe speed.
Chapter 5: Integration

- Revisit the velocity versus location issue.
- Velocity is the derivative of location.
- How could we get the location from first principles (not as an antiderivative)?