4.5 Indeterminate Forms

• We defined the derivative as

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

• The expression \( \frac{f(x+h) - f(x)}{h} \) is an indeterminate expression of the form \( \frac{0}{0} \).

• \( \frac{0}{0} \) is undefined, but it does make sense to think of the limit of a quotient whose numerator and denominator both approach zero.

• We used this idea to define derivatives, but now that we know derivatives we can turn this concept around and use derivatives to compute indeterminate expressions.

• The basic tool for this task is (see page 291, textbook)

The Rule of L'Hospital\(^1\) Suppose \( f \) and \( g \) are differentiable and \( h'(x) \neq 0 \) near \( a \) except possibly at \( a \). Suppose that

\[ \lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0. \]

Then

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

if the limit on the right exists.

\(^1\) There are several different spellings of L’Hospital. Guillaume de L'Hôpital, 1661-1704, learned the rule from his hired teacher Johann Bernoulli, and published it in 1696 in the first textbook on Calculus, Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes.
• Example:

\[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1
\]
Why?

• A proof of the Rule of L’Hospital is beyond our scope but here is a calculation that makes it plausible. Suppose $f(a) = g(a) = 0$.

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \frac{f(x) - f(a)}{x - a}
\]

\[
= \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

• The Rule of L’Hospital also applies if the limits of $f$ and $g$ are infinite, and if the limit of $f'/g'$ is plus or minus infinite. It also applies to one-sided limits and the case that $a$ is plus or minus infinity.
\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0 \frac{\infty}{\infty}
\]

\[
\lim_{x \to \infty} \frac{1}{x} = 0
\]

\[
\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1/x}{2x} = 0
\]

\[
\frac{1/x}{2x} = \frac{1}{2x^2}
\]
There are several other types of indeterminate expressions, i.e.,

\[
\frac{\infty}{\infty}, \quad 0 \times \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty.
\]

More examples:

\[
\lim_{x \to 0^+} x \times \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x}{1} = 0
\]
\[
\lim_{x \to 1} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right) = \\
= \lim_{x \to 1} \frac{x \ln x - (x-1)}{(x-1) \ln x} \\
= \lim_{x \to 1} \frac{\ln x + x \frac{1}{x} - 1}{\ln x + (x-1) \cdot \frac{1}{x}} \\
= \lim_{x \to 1} \frac{\ln x}{\ln x + \frac{x-1}{x}} \\
= \lim_{x \to 1} \frac{x \ln x}{x \ln x + x - 1} \\
= \lim_{x \to 1} \frac{\ln x + \frac{x}{x}}{\ln x + \frac{x}{x} + 1} \\
= \lim_{x \to 1} \frac{1 + \ln x}{2 + \ln x} \\
= \frac{1}{2}
\]
\[
\lim_{x \to 0} x^x = e^0 = 1
\]

\[
L = \ln x^x = x \ln x
\]

\[
\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{1/x}
\]

\[
= \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} \frac{x}{-1} = 0
\]
Figure 1. Graph of $y = x^x$.

\[
\frac{d}{dx} \ln x^x = \frac{d}{dx} x \ln x \\
= \ln x + 1 = 0 \\
\ln x = -1 \\
x = e^{-1}
\]
\[
\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x
\]

\[
L = \lim_{n \to \infty} \ln \left(1 + \frac{x}{n}\right)^n
\]

\[
= \lim_{n \to \infty} n \ln \left(1 + \frac{x}{n}\right)
\]

\[
= \lim_{n \to \infty} \frac{\ln \left(1 + \frac{x}{n}\right)}{1/n^2}
\]

\[
= \lim_{n \to \infty} \frac{x}{1 + \frac{x}{n}} = x
\]
Suppose $p$ is any non-zero polynomial. Compute

$$L = \lim_{x \to \infty} \frac{p(x)}{e^x} =$$

$$\deg p = 0 \quad L = 0$$

$$1 \quad p' = \text{constant}$$

$$\infty \quad \frac{d}{e^x} \to 0$$

$$\frac{p}{e^x} \Rightarrow \frac{p'}{e^x} \Rightarrow \frac{p''}{e^x} \Rightarrow \frac{p'''}{e^x} \Rightarrow \ldots \Rightarrow 0$$
What about

\[
\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}
\]

\[
= \lim_{x \to 0} \frac{-\sin x}{6x}
\]

\[
= \lim_{x \to 0} \frac{-\cos x}{6} = \frac{-1}{6}
\]
What about

\[ 2 = \lim_{x \to 0} \frac{1 + \cos x}{1 - \sin x} \]

What about

\[ \lim_{x \to \infty} x^x = \]
\[ V = \frac{\pi}{3} s^2 h \]

\[ r^2 = \frac{s^2}{3} + (h-r)^2 \]

\[ s^2 = r^2 - (h-r)^2 \]

\[ V = \frac{\pi}{3} (r^2 - (h-r)^2) \cdot h = V(h) \]

\[ = \frac{\pi}{3} \left( r^2 h - (h-r)^2 h \right) \]

\[ = \frac{\pi}{3} \left( r^2 h - h^3 + 2rh^2 - r^2 h \right) \]

\[ = \frac{\pi}{3} \left( -h^3 + 2rh^2 \right) \]

\[ V' = -3h^2 + 4rh = 0 \]

\[ = h(4r - 3h) \]

\[ h = \frac{4}{3} r \]

\[ V = \frac{\pi}{3} \left( -\frac{4}{27} r^3 + 2 \pi \cdot \frac{4}{3} r \right) \]
\[ 2\pi r = (2\pi - \Theta)R \]
\[ r = \frac{(2\pi - \Theta)}{2\pi} R \]

\[ h^2 = (R-r^2) - R - \frac{(2\pi - \Theta)^2 R^2}{4\pi^2} \]
\[ h^2 = \left(1 - \frac{(2\pi - \Theta)^2}{4\pi^2}\right) R^2 \]
\[ = \frac{4\pi^2 - 4\pi^2 + 4\pi\Theta - \Theta^2}{4\pi^2} R^2 \]
\[ = \frac{4\pi\Theta - \Theta^2}{4\pi^2} R^2 \]
\[ h = \frac{\sqrt{4\pi\Theta - \Theta^2}}{2\pi} R \]
\[ V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \left( \frac{2\pi - \theta}{2\pi} \right)^2 \left( \frac{1}{4\pi^2} \right) \sqrt{8\pi \theta - \theta^2} \frac{R^3}{2\pi} \]

\[ = \frac{1}{2^4\pi^2} (2\pi - \theta)^2 \sqrt{8\pi \theta - \theta^2} \]

\[ V' = 0 \quad \ldots \quad \theta = \frac{2\sqrt{2}}{3} \pi \approx 94.0 \]

\[ \ldots \]

\[ V = \frac{2\sqrt{3} \pi}{27} R^3 \]
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**Solve** \( V = \frac{\pi}{3} r^2 \sqrt{R^2 - r^2} \) for \( R \)

\[
R^2 = \pi^2 r^6 + 9V^2
\]
\[
R = \frac{\pi^2 r^6 + 9V^2}{\pi^2 r^2}
\]

\[
A = \frac{1}{2} r R
\]

\[
A' = 0 \quad \Rightarrow \quad r = \frac{1/3}{2 \pi^{1/3}}
\]

\[
h = \sqrt{R^2 - r^2} = \frac{6^{1/3}}{\pi^{1/3}} V^{1/3}
\]

\[
V = 3^{1/3} \quad \Rightarrow \quad V^{1/3} = 7
\]