More Minimization and Maximization

A very cool problem

It takes a certain power $P$ to keep a plane moving along at a speed $v$. The power needs to overcome air drag which increases as the speed increases, and it needs to keep the plane in the air which gets harder as the speed decreases. So assume the power required is given by

$$P = cv^2 + \frac{d}{v^2}, \quad c, d > 0$$

where $c$ and $d$ are positive constants. (They depend on your plane, your altitude, and the weather, among other things.) What is the choice of $v$ that will minimize the power required to keep moving at speed $v$? Suppose you have a certain amount of fuel and the fuel flow required to deliver a certain power is proportional to that power. What is the speed $v$ that will maximize your range (i.e., the distance you can travel at that speed before your fuel runs out)? Finally, what is the ratio of the speed that maximizes the distance and the speed that minimizes the required power? Do you notice anything remarkable? Do you have any conclusions that are relevant to flying a real plane? Check with a pilot friend if the conclusion is valid.
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\[ V = \frac{\pi r^2 h}{3} \]

\[ \frac{r}{H-h} = \frac{R}{H} \]

\[ r = \frac{R}{H} \cdot (H-h) \]
\[ P(u) = cu^2 + \frac{d}{u^2} \]

\[ p'(u) = 2cu - 2d \frac{u}{u^3} = 0 \]

\[ 2cu = \frac{2d}{u^3} \]

\[ 2cu^4 = 2d \]

\[ u^4 = \frac{d}{c} \]

\[ \sqrt[4]{u} = 2c \]

\[ \sqrt[4]{u} = \sqrt[4]{d/c} = \frac{d}{\sqrt[4]{c^7}} = \frac{d^{1/4}}{\sqrt[4]{c}} = \sqrt[4]{\frac{d}{c}} \]

\[ \nu = \sqrt[4]{\frac{d}{c}} \]

\[ R = \sqrt[4]{\frac{d}{c}} \]

\[ \frac{V}{\nu} = \frac{V}{\sqrt[4]{\frac{d}{c}}} = \max \]

\[ T = \frac{k_T}{P} \]

\[ k_V = \frac{1}{P} \]

\[ k_T = 1 \]
\[ R(v) = \frac{v}{P} = \frac{v}{cv^2 + \frac{d}{v^2}} = \frac{v^3}{cv^4 + d} \]

\[ R'(v) = \frac{3v^2(cv^4 + d) - v^34cv^3}{(cv^4 + d)^2} \]
\[ = \frac{3cv^6 + 3d - 4cv^6}{(cv^4 + d)^2} \]
\[ = \frac{3d - cv^6}{(cv^4 + d)^2} \]
\[ = 0 \]
\[ 3dv^2 - cv^6 = 0 \]
\[ \frac{3d}{cv^4} = 0 \]
\[ cv^4 = 3d \]
\[ v^4 = \frac{3d}{c^4} \]
\[ v = \sqrt[4]{\frac{3d}{c^4}} \]

\[ 3 \sqrt[4]{v} = \sqrt[4]{v^2} = \sqrt[4]{(\frac{3d}{c})^2} \]
The illumination at a point is inversely proportional to the square of the distance of the point from the light source and directly proportional to the intensity of the light source. Suppose two light sources are $s$ feet apart and their intensities are $I$ and $J$, respectively. Where is the point $P$ on the line connecting the two light sources at which the sum of its illumination from the two light sources is a minimum.

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2\bar{I} (s-x)^3 = 2\bar{J} x^3 \quad \bar{J} = \frac{1}{2} \\
\bar{I} (s-x)^3 = \bar{J} x^3 \\
\bar{I} \left( \frac{x^3}{(s-x)^3} \right) = \bar{J} \left( \frac{1}{(s-x)^3} \right) ^\frac{1}{3} \\
\bar{I} \left( \frac{x}{s-x} \right) ^\frac{1}{3} = \bar{J} \left( \frac{1}{(s-x)^3} \right) ^\frac{1}{3} = \alpha \\
\frac{x}{s-x} = \left( \frac{\bar{I}}{\bar{J}} \right) ^\frac{1}{3} = \alpha \\
\int_0^x (s-x) dx = \alpha \int_0^x (s-x) \\
x = \alpha (s-x) \\
\frac{x + \alpha x}{1 + \alpha} = \frac{\alpha s}{1 + \alpha} \\
\int_0^x ds = \frac{\bar{I} \frac{1}{3}}{\bar{J} \frac{1}{3}} \\
x = \frac{\bar{I} \frac{1}{3}}{\bar{J} \frac{1}{3} + \bar{J} \frac{1}{3}} s
If time remains revisit

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1. \quad \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{x} = 1
\]

Figure 1. Graphs of \( x \) and \( \sin x \).