Math 1310-4  Notes of October 21, 2019

• The plan for this week is to collect a couple more tools for analyzing extreme values, and then spend the rest of the week solving word problems. It will be fun!

• **The Mean Value Theorem** (page 272, textbook) If $f$ is a differentiable function on the interval $[a, b]$ then there exists a number $c$ in the interval $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or, equivalently

$$f(b) - f(a) = f'(c)(b - a).$$

This statement is deceptively simple, like the Intermediate Value Theorem, but it’s very powerful.

![Figure 1. The Mean Value Theorem.](image-url)
• Example $f(x) = x^2$, $0 \leq x \leq 1$

$\begin{align*}
\frac{x - 0}{1 - 0} &= m = 1 \\
\frac{1}{1} &= 1
\end{align*}$

$f(x) = x^2$

$f'(x) = 2x = 1$

$x = \frac{1}{2}$

**Figure 2.** $f(x) = x^2$, $0 \leq x \leq 1$. 
• Example \( f(x) = x^3, \ 0 \leq x \leq 1 \)

\[ f'(x) = \frac{2}{3x^3} \]

\[ f'(x) = 3x^2 = 1 \]

\[ x^2 = \frac{1}{3} \]

\[ x = \frac{1}{\sqrt{3}} > \frac{1}{2} \]

**Figure 3.** \( f(x) = x^3, \ 0 \leq x \leq 1 \).
• Recall our previous discussion of derivatives:

• If \( f'(x) > 0 \) on an interval then \( f \) is increasing on that interval.

• If \( f'(x) < 0 \) on an interval then \( f \) is decreasing on that interval.

• Recall that a critical number of a function \( f \) is a number \( c \) where \( f'(c) = 0 \) or \( f'(c) \) does not exist.

• **The First Derivative Test.** Suppose that \( c \) is a critical number of a continuous function.

  • if \( f' \) changes from positive to negative at \( c \) then \( f \) has a local maximum at \( c \). (First we go up, then we go down.)

  • if \( f' \) changes from negative to positive at \( c \) then \( f \) has a local minimum at \( c \). (First we go down, then we go up.)

• Example: \( f(x) = x^2 \)

\[
\begin{align*}
f'(0) &= 0 \\
f(0) &= 0 \\
f'(x) &= 2x \\
f(x) &= \begin{cases} x & x > 0 \\
0 & x = 0 \\
-x & x < 0 \end{cases}
\end{align*}
\]
- \( f(x) = \sin x \)

- \( f(x) = |x| \)

- \( f(x) = \sqrt{|x|} \)

\begin{align*}
\text{if} & \quad x < 0 \\
\frac{d}{dx} f(x) & = \left(-x\right)^{1/2} \\
\frac{d}{dx} f(x) & = \frac{-1}{2} \left(-x\right)^{-1/2} \\
& = -\frac{1}{2 \sqrt{-x}}
\end{align*}

\begin{align*}
\text{if} & \quad x > 0 \\
\frac{d}{dx} f(x) & = \frac{1}{2 \sqrt{x}} \\
& = \frac{-1}{2 \sqrt{-x}}
\end{align*}

\textbf{Figure 4.} \( f(x) = \sin x, \ 0 \leq x \leq 1. \)
• Clearly, if $f''$ is positive, then $f'$ is increasing. We are making a left turn as we drive from left to right.

• Similarly, if $f''$ is negative we make a right turn.

• This leads to the following definition: A function (or its graph) is concave up on an interval $I$ if $f'$ is an increasing function on $I$. In is concave down if $f'$ is a decreasing function on $I$.

• If $f''$ is positive on $I$ then $f$ is concave up on $I$, and if $f''$ is negative, then $f$ is concave down. (The textbook calls this the “concavity test”, see page 275.)

• Example: $f(x) = x^2$

• Example: $f(x) = \sin x$
Inflection Points

- A point on the graph of $f$ at which the direction of concavity changes (from up to down or down to up) is called an \textbf{inflection point}.

- If $f$ is twice differentiable the second derivative changes sign.

- If the second derivative is continuous the second derivative equals zero at the inflection point.

- If the second derivative is zero there may or may not be an inflection point.

- Example: $f(x) = x^3$

- Example: $f(x) = x^4$
• Second derivatives can be used to identify the nature of a stationary point.

• The textbook (p. 275) introduces the **second derivative test**.

• Suppose $f''$ is continuous near $c$. Then
  
  (a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.
  
  (b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.
  
  (c) If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

• **Examples**
  
  $f(x) = x^2$
\[ f(x) = \sin x \]

\[ f(x) = x^3 \]

\[ f(x) = x^4 \]
\[ f(x) = \frac{1}{x^2 + 1} \]
\[ f'(x) = -\frac{2x}{(x^2 + 1)^2} \]
\[ f''(x) = \frac{6x^2 - 2}{(x^2 + 1)^3} \]

\[ 6x^2 - 2 = 0 \]
\[ x^2 = \frac{1}{3} \]
\[ x = \pm \frac{1}{\sqrt{3}} \]

Figure 5. \( f(x) = \frac{1}{x^2 + 1} \).
# 1

\[ f(x) = e^{-10x} - e^{-3x} \quad \left[ 0, 2 \right] \quad f(0) = 0 \]

\[ f'(x) = -10 e^{-10x} + 3 e^{-3x} = 0 \]

\[ 10 e^{-10x} = 3 e^{-3x} \quad -0.0024 \]

\[ \frac{10}{3} = \frac{e^{-3x}}{e^{-10x}} = e^{7x} \]

\[ 7x = \ln \left( \frac{10}{3} \right) \]

\[ x = \frac{1}{7} \ln \left( \frac{10}{3} \right) = 0.172 \]

\[ f = -0.4178 \]

# 6

\[ y = x^3 + 10x \]

\[ x = x(t) \]

\[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \]

\[ 22 \cdot 5 = 110 \]

\[ \frac{dy}{dx} = 3x^2 + 10 \quad x = 2 \Rightarrow \frac{dy}{dx} = 22 \]
$s' = \frac{s}{h} = \frac{d+s}{H}$

$\frac{s'h - s'h'}{h^2} = \frac{s'}{H}$

$\frac{s'}{h} - \frac{s'h'}{h^2} = \frac{s'}{H}$

$\frac{s'}{h} - \frac{s'h'}{h^2} = \frac{s'h'}{h^2}$

$s'(H-h) = s'H - s'h = \frac{s'h'H}{h}$

$s' = \frac{s'h'H}{h(H-h)} = \frac{(d+s)h'H}{H(H-h)}$
\[ S = 0 \quad h = 0 \quad = \frac{d H - H}{H^2} \]

\[ = \frac{d H}{H} \]

\[ = \frac{d v}{H} \]