Announcements

- Exam 2 will be on differentiation rules. It will take place on Friday, October 18, in HEB 2004.

- Wednesday will be a review session. Please read the notes of tomorrow before class and make a list of questions. On Wednesday I will assume you have read the notes and refreshed your memory, and will answer questions.
• Today: more on Related Rates, beginning of Minimization and Maximization.

• Example:

1, page 256, textbook. You are blowing air into a spherical balloon at the rate of 100 cubic centimeters liter per second. How fast (measured in square centimeters per second) is the surface area of the balloon growing when the diameter of the balloon is 50 cm?

• There is a tradition that textbook examples contain specific numbers, like 100, or 50.

• However, it may be advantageous to use variables, for several reasons:
  – If you have a similar problem with different numbers you don’t have to redo the calculation. Just substitute the new numbers.
  – In other words, the result of your calculation is a formula, rather than a number.
  – A more important reason, however, is that variables carry information that disappears when using numbers. For example, you can check the units of your quantities at any time in the calculation.

• So let’s do the above problem in general. Suppose $V$ denotes the volume, $S$ denotes the surface area, $d$ denotes the diameter, $r$ denotes the radius, and $V'$ denotes the rate at which we are blowing air into the balloon.

• We want a formula that gives us $r'$ as a function of $d$ and $V'$. (It’s a little contrived to have the increase in the diameter as a function of the radius, but let’s indulge the author of this problem.)

• At the end, if we want to, we can substitute numbers (and perhaps compare our answer with the answer in the book, which is $1/(25\pi)$ cm per second).

$$r' = f(d, V')$$
\[ V = \frac{4}{3} \pi r^3 \]

\[ V' = \frac{4}{3} \pi 3r^2 r' = 4\pi r^2 r' \]

\[ r' = \frac{V'}{4\pi r^2} \]

\[ d = 2r \]
\[ 4r^2 = d^2 \]

\[ \frac{V'}{\pi d^2} = \frac{cm^3/s}{cm^2} = \frac{cm}{s} \]

\[ d = 50 \quad V' = 100 \]

\[ r' = \frac{100}{\pi 50^2} = \frac{1}{25\pi} \]
4.2 Minimum and Maximum Values

• This is a major theme in applications of the derivative.

• Clearly we want to be able to tell when or where something is as large or as small as possible.

• Example: A rock is thrown upwards, what is its maximum height?

• maximum height occurs when vertical velocity is zero.

\[ h(t) = -\frac{g}{2} t^2 + v_0 t + h_0 \]

\[ v(t) = h'(t) = -gt + v_0 = 0 \]

\[ t = \frac{v_0}{g} \]

\[ h\left(\frac{v_0}{g}\right) = -\frac{g}{2} \left(\frac{v_0}{g}\right)^2 + v_0 \frac{v_0}{g} + h_0 \]

\[ h_{\text{max}} = \frac{1}{2} \frac{v_0^2}{g} + h_0 \]

\[ m \quad \frac{(m/s)^2}{m/s^2} + m \]

\[ \frac{m^2}{s^2} = m \]

• Basic Theme When a function is minimized or maximized its derivative is zero. (plus some caveats ... see below.)

• We start by building a suitable vocabulary.
Example.

\[ f(x) = 2.2|x| + \frac{e^x}{20x^2 + 1}. \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fancy_graph.png}
\caption{A fancy graph.}
\end{figure}
• Definition 1, page 262: Let \( c \) be a number in the domain \( D \) of a function \( f \). Then \( f(c) \) is the **absolute maximum** value of \( f \) on \( F \) if \( f(c) \geq f(x) \) for all \( x \) in \( D \). Similarly \( f(c) \) is the **absolute minimum** value of \( f \) on \( F \) if \( f(c) \leq f(x) \) for all \( x \) in \( D \). Collectively the minimum and maximum values of \( f \) are called the **(absolute) extreme values** of \( f \).

• A common alternative to the word **absolute** is the word **global**. Note that in this context the word **absolute** has nothing to do with the absolute value of something.

• The number \( f(c) \) is a **local maximum** value of \( f \) if \( f(c) \geq f(x) \) when \( x \) is near \( c \). Similarly, \( f(c) \) is a **local minimum** value of \( f \) if \( f(c) \leq f(x) \) when \( x \) is near \( c \).

• When we say that something is true for \( x \) near \( c \) we mean that it is true on an arbitrarily small open interval containing \( c \). (To be rigorous, we need to modify this definition slightly if \( c \) is a boundary point of the domain of \( f \). This is a good exercise, think about it!)

• A common alternative to the word **local** is the word **relative**.

• Every absolute extreme value is also a local extreme value.

• It seems intuitively clear, and is proved in the textbook, that \( f(c) \) can be a (local or absolute) extreme value only in three cases:
  1. \( f'(c) = 0 \)
  2. \( f'(c) \) does not exist
  3. \( c \) is a boundary point of the domain of \( f \).

• Definition 5, p. 266: A **critical number** of a function \( f \) is a number \( c \) in the domain of \( f \) such that \( f'(c) = 0 \) or \( f'(c) \) does not exist.

• A number \( c \) where \( f'(c) = 0 \) is also called a **stationary point**.

• A number \( c \) where \( f'(c) \) does not exist is also called a **singular point**.

• In the literature, instead of the phrase “critical number” you might also see “critical point”, and the phrase may or may not include endpoints of intervals.

• Note that the phrase “extreme value” refers to numbers in the range of \( f \) whereas the phrase “critical number” refers to numbers in the domain of \( f \).
• The Closed Interval Method for finding extreme values of $f$ on a closed interval $[a, b]$ now proceeds as follows (modified from page 266):

1. Find all critical numbers.
2. Evaluate $f$ at the endpoints of $[a, b]$.
3. The largest function value so found is the absolute maximum value, the smallest is the absolute minimum value.
4. To find relative extreme value examine what is happening near the critical numbers and the endpoints.

• In a nutshell: To find extreme values find and examine all endpoints, all numbers $c$ where $f'(c) = 0$, and all numbers $c$ where $f'(c)$ does not exist.

• More succinctly: Extreme values can only occur at stationary points, singular points, and endpoints.

• Stationary points constitute the most important case.

• Let’s look at some examples:

• $f(x) = x^2$.

\[
\begin{align*}
\frac{df}{dx} &= 2x = 0 \\
x &= 0
\end{align*}
\]
$f(x) = \cos x$

$f'(x) = -\sin x$

$f(x) = x^3$

$f'(x) = 3x^2$

$x = 0$

$f(x) = e^x$

$f(x) = |x|$

$f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$f(x) = \frac{1}{1+x^2}$, $-\infty < x < 1$

$f'(x) = \frac{-2x}{(1+x^2)^2}$

$x > 0$
Recall Home Work Problem: $f(x) = x^2(1 - x)$.  

\[
\begin{align*}
  f(x) &= x^2 - x^3 \\
  \frac{df}{dx} &= 2x - 3x^2 \\
  &= x(2 - 3x) = 0 \\
  x &= \frac{2}{3} \\
  f(\frac{2}{3}) &= \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27}
\end{align*}
\]

**Figure 2.** An easy problem.
Q & A

#7  \( f(x) = x^{100} \)

\( f^{(100)}(x) = 100 ! \)

\( f^{(100)}(x) = 100 ! \)  \( f^{(100)}(x) = 100 ! \)

\( f(x) = x^1 \)  \( f'(x) = 1 \)
\( f(x) = x^2 \)  \( f''(x) = 2x \)
\( f(x) = x^3 \)  \( f'''(x) = 3x^2 \rightarrow 6x \)
\( f(x) = x^4 \)  \( f^{(IV)}(x) = 4x^3 \rightarrow 12x^2 \rightarrow 24x \rightarrow (24) \)

120
720
5040

\( f(x) = x^n \)  \( f^{(n)}(x) = n! \)

#30

\( f(x) = e^x \)

\( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \)
\[ f(x) = 10(x-1)(x-2)(x-3) - 1 \]

\[ x_{n+1} = x_n - \frac{10x_n^3 - 60x_n^2 + 110x_n - 61}{30x_n^2 - 120x_n + 110} \]

\begin{tabular}{|c|c|}
\hline
n & \( x_n \) \\
\hline
0 & 1 \\
1 & 1.05 \\
2 & 1.0543 \\
3 & 1.05435072 \\
4 & \\
\hline
\end{tabular}

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\[ x^2 + xy + y^2 = 9 \quad \frac{dy}{dx} = 1 \]

\[ 2xx' + x'y + x + 2y = 0 \]

\[ x'(2x+2y) = -x - 2y \]

\[ x' = - \frac{x+2y}{2x+y} = 0 \]

\[ y = -\frac{x}{2} \]

\[ x = x(y) \]

\[ x^2 - \frac{x^2}{2} + \frac{x^2}{4} = 9 \]

\[ \frac{3}{4} x = 9 \]
\[ x^2 = \frac{9 \cdot 4}{3} = 12 \]
\[ x = \frac{\pm \sqrt{12}}{2} \]
\[ y = \frac{\sqrt{12}}{2} \]

**Linear Approximation**

\[ u(x) = f(g(x)) \]
\[ u'(x) = f'(g(x)) \cdot g'(x) \]

\[ f(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]
\[ \Delta y = f(x + \Delta x) - f(x) \]
\[ \Delta x = (x + \Delta x) - x \]

\[ f(x) \approx \frac{\Delta y}{\Delta x} \]

\[ \Delta y \approx f'(x) \Delta x \]
$e^{0.01}$

$f(x) = e^x$

$e^0 = 1$

\[ e^{-e} \approx (0.01-1) \]

$e^{0.01} \approx e^0 + (0.01-1)$

$= 1 + 0.01$

$= 1.01$

\[ f(x) = e^x \]

\[ p(x) = a x^3 + b x^2 + c x + d \]

$p = \frac{\partial^2}{\partial x^2} (a x^2 + b x + c)$

$p'' = 6a x + 2b$

$p''' = 6a$

$f(1) = e^1 = p(1) = d$

$d = 1$

$f'(1) = 1 = p'(1) = c$

$c = 1$

$f''(1) = 1 = p''(1) = 2b$

$b = 1/2$

$f'''(1) = 1 = p'''(1) = 6a$

$a = 1/6$