Today: more on Related Rates, beginning of Minimization and Maximization.

Example:
You are blowing air into a spherical balloon at the rate of 1 liter per second. How fast (measured in square centimeters per second) is the surface area of the balloon growing when the radius of the balloon is 10 cm. (Note: a liter contains 1000 cubic centimeters.)
4.2 Minimum and Maximum Values

• This is a major theme in applications of the derivative.
• Clearly we want to be able to tell when or where something is a large or as small as possible.
• Example: A rock is thrown upwards, what is its maximum height?
• maximum height occurs when vertical velocity is zero.

$$h(t) = -\frac{g}{2}t^2 + v_0 t + h_0$$

• Basic Theme When a function is minimized or maximized its derivative is zero. (plus some caveats ... see below.)

• We start by building a suitable vocabulary.
- Example.

\[ f(x) = 2.2|x| + \frac{e^x}{20x^2 + 1}. \]

**Figure 1.** A fancy graph.
• Definition 1, page 262: Let $c$ be a number in the domain $D$ of a function $f$. Then $f(c)$ is the **absolute maximum** value of $f$ on $F$ if $f(c) \geq f(x)$ for all $x$ in $D$. Similarly $f(xc)$ is the **absolute minimum** value of $f$ on $F$ if $f(c) \leq f(x)$ for all $x$ in $D$. Collectively the minimum and maximum values of $f$ are called the **extreme values** of $f$.

• A common alternative to the word absolute is the word **global**. Note that in this context the word absolute has nothing to do with the absolute value of something.

• The number $f(c)$ is a **local maximum** value of $f$ if $f(c) \geq f(x)$ when $x$ is near $c$. Similarly, $f(c)$ is a **local minimum** value of $f$ if $f(c) \leq f(x)$ when $x$ is near $c$.

• When we say that something is true for $x$ near $c$ we mean that it is true on an arbitrarily small open interval containing $c$. (To be rigorous, we need to modify this definition slightly if $c$ is a boundary point of the domain of $f$. This is a good exercise, think about it!)

• A common alternative to the word local is the word **relative**.

• Every absolute extreme value is also a local extreme value.

• It seems intuitively clear, and is proved in the textbook, that $f(c)$ can be a (local or absolute) extreme value only in three cases:
  1. $f'(c) = 0$
  2. $f'(c)$ does not exist
  3. $c$ is a boundary point of the domain of $f$.

• Definition 5, p. 266: A **critical number** of a function $f$ is a number $c$ in the domain of $f$ such that $f'(c) = 0$ or $f'(c)$ does not exist.

• A number $c$ where $f'(c) = 0$ is also called a **stationary point**.

• A number $c$ where $f'(c)$ does not exist is also called a **singular point**.

• Instead of the phrase “critical number” you might also see “critical point”, and the phrase may or may not include endpoints of intervals.

• Note that the phrase “extreme value” refers to numbers in the range of $f$ whereas the phrase “critical number” refers to numbers in the domain of $f$.

• **The Closed Interval Method** for finding extreme values of $f$ on
a closed interval \([a, b]\) now proceeds as follows (modified from page 266):

1. Find all critical numbers.
2. Evaluate \(f\) at the endpoints of \([a, b]\).
3. The largest function value so found is the absolute maximum value, the smallest is the absolute minimum value.
4. To find relative extreme value examine what is happening near the critical numbers and the endpoints.

- **In a nutshell:** To find extreme values find and examine all endpoints, all numbers \(c\) where \(f'(c) = 0\), and all numbers \(c\) where \(f'(c)\) does not exist.

- More succinctly: Extreme values can only occur at stationary points, singular points, and endpoints.

- Stationary points constitute the most important case.

- Let’s look at some examples:

- \(f(x) = x^2\).
\[ f(x) = \cos x \]

\[ f(x) = x^3 \]

\[ f(x) = e^x \]

\[ f(x) = |x| \]

\[ f(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \]

\[ f(x) = \frac{1}{1+x^2}, \quad -\infty < x \leq 1 \]
Recall Home Work Problem: \( f(x) = x^2(1 - x) \).

\[ \text{Figure 2. An easy problem.} \]