3.9 Linear Approximation and Differentials

- **Major Idea:** The tangent is the best local linear approximation of a function.

\[ f(x) \approx L(x) = f(a) + f'(a)(x - a) \]

- The graph of \( L \) is the tangent of the graph of \( f \) at \( x = a \) and \( L \) is the linearization of \( f \) at \( a \).

\[ f(x) \approx L(x) = f(a) + f'(a)(x - a) \]

- Revisit

\[ f(x) \approx L(x) = f(a) + f'(a)(x - a) \]

- The graph of \( L \) is the tangent line at \( x = a \).

\[ a = L(a) = f(a) - f'(a)(a - a) = f(a) \]

\[ f'(a) = L'(a) = \]
Figure 1. Linearization of the exponential at $x = 0$.

Example:
\[ e^x \approx L(x) = x + 1 \]

\[
\begin{array}{ccccc}
  x   & e^x & x + 1 & e^x - (x + 1) \\
  0.2 & 1.221 & 1.2 & 0.021 \\
  0.1 & 1.105 & 1.1 & 0.005 \\
  0.0 & 1 & 1 & 0.0 \\
  -0.1 & 0.95 & 0.9 & 0.005 \\
  -0.2 & 0.819 & 0.8 & 0.019 \\
\end{array}
\]
Figure 2. graph of $e^x - (x + 1)$. 
Differentials

Recall

$$\Delta y = f(a + \Delta x) - f(a)$$

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

• If $dx$ and $dy$ were variables this would lead to

$$dy = f'(a)dx \approx \Delta y$$ \hspace{1cm} (1)

and

$$f(x) = f(a + \Delta x) = f(a) + \Delta y \approx f(a) + f'(a)dx$$

• In the equation (1) $dx$ and $dy$ are in fact often considered variables. If so they are called differentials.

• Another way of looking at linear approximation is

$$\Delta y \approx dy = f'(a)dx.$$ 

• The derivative is the limit of the quotient of change in $f$ divided by the change in $x$. Correspondingly:

$$\Delta y \approx dy = f'(a)dx.$$ 

The change in $f$ equals approximately the change in $x$ multiplied with the derivative.
The derivative gives you the factor by which to multiply the change in the independent variable to get the change in the dependent variable.

- these statements are variations of our theme that the derivatives measures the rate of change of the function.

**Example:**

Approximate $\sqrt{26}$ using the fact that $f(x) = \sqrt{x}$

\[ \sqrt{25} = 5 \quad \text{and} \quad \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \]

- The change in $x$ is 1. The derivative of $f$ at $x = 25$ equals

\[ \frac{1}{2\sqrt{25}} = \frac{1}{10}. \]

The square root of 26 equals approximately 5.1.

\[ 5.09902 = \sqrt{26} \approx \sqrt{25} + \frac{1}{10} \times 1 = 5.1. \]
Example 4, p. 244, textbook: The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. Approximate the maximum error in using this value of the radius to compute the volume of the sphere.

Let \( r \) denote the radius of the sphere. Then its volume is

\[
V = \frac{4}{3}\pi r^3.
\]

Denoting the error in \( r \) by \( dr = \Delta r \), the corresponding error in \( V \) is

\[
\Delta V \approx dV = \frac{d}{dr}Vdr = 4\pi r^2 dr.
\]

With \( r = 21 \) and \( dr = 0.05 \) this becomes

\[
dV = 4\pi(21)^2 \times 0.05 = 277.09
\]

The maximum error in the volume is approximately 277 cubic centimeters.

That may seem a lot, but compare it with the relative maximum error which is

\[
\frac{dV}{V} = \frac{4\pi(21)^2 \times 0.05}{\frac{4}{3}\pi \times 21^3} \approx 0.0071 = 0.71\%.
\]

Compare this with the error of

\[
\frac{0.05}{21} = 0.0024 = 0.24\%
\]

in \( r \).
Error and Relative Error

- Example: Making cylinders

\[ V = \pi r^2 h \]

- Denote errors in \( v, r, \) and \( h \) by \( \Delta V, \Delta r, \) and \( \Delta h, \) respectively. The corresponding relative errors are \( \Delta V/V, \Delta r/r, \) and \( \Delta h/h, \) respectively.

- How does the relative error in \( V \) depend on the relative error in \( r \) or \( h? \)

- Is it more important to get the radius right, or the height? Does the answer depend on the numerical values of radius and height?

- We get

\[
\begin{align*}
\text{h:} & \quad \frac{\Delta V}{V} \approx \frac{dV}{dh} \frac{dh}{V} = \frac{\pi r^2 dh}{\pi r^2 h} = \frac{dh}{h} \\
\text{and} \quad \text{r:} & \quad \frac{\Delta V}{V} \approx \frac{dV}{dr} \frac{dr}{V} = \frac{2\pi rh dh}{\pi r^2 h} = \frac{2dr}{r}
\end{align*}
\]

- An error of 1% in the radius will cause twice as much relative error in the volume as an error of 1% in the height.

- It’s twice as important to get the radius right as it is to get the height right.
Newton’s Method

• A major idea! Linearize, solve the linear problem, repeat.

• Deliberately introduced in hw problems (with explanation). You want to learn how to learn concepts on your own by reading . . .

• We’ll visit the subject in more depth on October 29.

• However, here is the basic idea. We want to solve $f(x) = 0$. $x$ is a “root”, “zero”, or “$x$-intercept” of $f$.

• Suppose we can’t. For example: $f(x) = e^x + x = 0$. Try it!

• Start with an approximation. Linearize. Solve linear problem. Repeat.

• Suppose $x_k$ is the current approximation.

• Solve

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) = 0$$

for $x = x_k - f(x_k)/f'(x_k)$.

• Thus Newton’s Method is defined by:

1. $x_0$ given. (Its value depends on the problem.)
2. Define

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
for $k = 0, 1, 2, \ldots$

3. Stop when the newest $x_{k+1}$ is good enough. (The meaning of that phrase also depends on the problem.)
Newton's Method graphically:

Example \( \sqrt[3]{27} = 3 \)

\[ f(x) = x^3 - 2 = 0 \]

\[ f'(x) = 3x^2 \]

\[ x_0 = 1.2 \]

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

\[ = x_k - \frac{x_k^3 - 2}{3x_k^2} \]

\[ = \frac{3x_k^3 - x_k^3 + 2}{3x_k^2} \]

\[ = \frac{2x_k^3 + 2}{3x_k^2} \]
Return to the linear approximation of the exponential. It matches the value and the derivative of the exponential at $x = 0$. Could we match second derivatives (in addition to value and first)? third? $n$-th? ("Laboratory Project", page 247).

**Figure 3.** Matching more derivatives.
How can we do this?