Review of differentiation rules.

Again, $c$ is a constant, $f$ and $g$ are functions, a prime denotes differentiation.

\[
\frac{d}{dx} x^r = r x^{r-1} \quad \text{Power Rule}
\]
\[
\frac{d}{dx} \sin x = \cos x \quad \text{sin}
\]
\[
\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \text{arcsin}
\]
\[
\frac{d}{dx} \cos x = -\sin x \quad \text{cos}
\]
\[
\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}} \quad \text{arccos}
\]
\[
\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} \quad \text{tan}
\]
\[
\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \quad \text{arctan}
\]
\[
\frac{d}{dx} e^x = e^x \quad \text{exponential}
\]
\[
\frac{d}{dx} \ln x = \frac{1}{x} \quad \text{logarithm}
\]
\[
(cf)' = cf' \quad \text{Constant Factor Rule}
\]
\[
(f \pm g)' = f' \pm g' \quad \text{Sum and Difference Rules}
\]
\[
(fg)' = f'g + fg' \quad \text{Product Rule}
\]
\[
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \text{Quotient Rule}
\]
\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \quad \text{Chain Rule}
\]
Derivative of an inverse function

• Recall how we computed the derivative of the logarithm:

\[ e^{\ln x} = x \implies \left( \frac{d}{dx} \ln x \right) e^{\ln x} = \left( \frac{d}{dx} \ln x \right) x = 1 \]

\[ \implies \frac{d}{dx} \ln x = \frac{1}{x}. \]

• The idea can be applied to a general inverse function:

Differentiating in

\[ f\left( f^{-1}(x) \right) = x \]

gives

\[ f'\left( f^{-1}(x) \right) \left( \frac{d}{dx} f^{-1}(x) \right) = 1 \]

which gives

\[ \left( \frac{d}{dx} f^{-1}(x) \right) = \frac{1}{f'(f^{-1}(x))} \]

• Thus the derivative of the inverse is the reciprocal of the derivative of the original function.

• This simple statement gets complicated by the fact that the two derivatives are not evaluated at the same point. The argument of the original derivative is in the domain of \( f \) and the argument of the derivative of the inverse is in the domain of the inverse, naturally.

• It is not worth memorizing this formula, however. If you ever need a formula for the derivative of an inverse apply the technique directly to that function, as we did for the logarithm, and the inverse trig functions.
The derivative of the inverse sometimes gets confused with the slope of a pair of perpendicular lines. Two perpendicular lines have slopes that are **negative reciprocals** of each other. A function and its inverse have derivatives that are **reciprocals with the same sign**.

- **Exercise:** Can you name some functions that equal its inverse? (Such functions of course have the **same** derivatives as their inverses.)
The Derivative of

\[ f(x) = \log_a(x) \]

\[ \frac{d}{dx} \log_a(x) = \frac{1}{x \ln a} \]
Example 5, p. 222 textbook. Compute the derivative of

\[ f(x) = \ln \frac{x + 1}{\sqrt{x - 2}} \]

in two different ways.

\[ f'(x) = \frac{\sqrt{x - 2}}{x + 1} \cdot \frac{d}{dx} \frac{x + 1}{\sqrt{x - 2}} \]

\[ = \frac{\sqrt{x - 2}}{x + 1} \cdot \frac{1}{2\sqrt{x - 2}} - \frac{(x + 1)}{2\sqrt{x - 2}} \]

\[ = \frac{x - 2 - (x + 1)}{2(x + 1)(x - 2)} \]

\[ = \frac{2x - 4 - (x + 1)}{2(x + 1)(x - 2)} \]

\[ = \frac{x - 5}{2(x + 1)(x - 2)} \]
\[ f(x) = \ln \frac{x+1}{\sqrt{x-2}} \]

\[ = \ln(x+1) - \ln \sqrt{x-2} \]

\[ = \ln(x+1) - \frac{1}{2} \ln(x-2) \]

\[ f'(x) = \frac{1}{x+1} - \frac{1}{2} \frac{1}{x-2} \]

\[ = \frac{2(x-2) - (x+1)}{2(x+1)(x-2)} \]

\[ = \frac{x-5}{2(x+1)(x-2)} \]

\[ y = \ln \frac{x+1}{\sqrt{x-2}} \]

\[ e^y = \frac{x+1}{\sqrt{x-2}} \]

\[ ye^y = \frac{\sqrt{x-2} - (x+1)}{2\sqrt{x-2}x-2} \]

\[ y' = \frac{\sqrt{x-2} - (x+1)}{2\sqrt{x-2}x-2} \cdot \frac{\sqrt{x-2}}{x+1} \]

\[ = \frac{x-2 - \frac{1}{2}(x+1)}{(x-2)(x+1)} \]

\[ = \frac{\frac{1}{2}x - \frac{5}{2}}{(x-2)(x+1)} \]

\[ = \frac{x-5}{2(x-2)(x+1)} \]
The derivative of 
\[ f(x) = x^x. \]

\[ \gamma = x^x \]

\[ \ln \gamma = x \ln x \]

\[ \frac{\gamma'}{\gamma} = 1 \cdot \ln x + x \cdot \frac{1}{x} \]

\[ = \ln x + 1 \]

\[ \gamma' = \gamma (\ln x + 1) = x^x (\ln x + 1) \]

\[ x^x \Rightarrow (\ln x)^x \]

\[ x^x \Rightarrow x \cdot x^{x-1} = x^x \]

\[ \frac{d}{dx} x^x = x^x (1 + \ln x) \]
The derivative of 
\[ f(x) = x^{x^x}. \]

\[ \frac{d}{dx} x^{x^x} = x^{x^x} \ln x + x^x \cdot \frac{1}{x} \]

\[ = x^{x^x} (1 + \ln x) \cdot \ln x + x^{x^x} \cdot \frac{1}{x} \]

\[ = x^{x^x} \left( x^x (1 + \ln x) \ln x + x^{x^x-1} \right) \]
The definition of $e$ as a limit

• (See page 224, textbook). The **power rule** for real (possibly irrational) exponents:

$$f(x) = x^r$$

(we have only shown it for rational exponents)

$$y = x^r$$

$$\ln y = r \ln x$$

$$\frac{y'}{y} = \frac{r}{x}$$

$$y' = \frac{r}{x} y = \frac{r}{x} x^r = r x^{r-1}$$
Derive the quotient rule by logarithmic differentiation.

\[ y = \frac{u}{v} \]

\[ \ln y = \ln u - \ln v \]

\[ \frac{y'}{y} = \frac{u'}{u} - \frac{v'}{v} \]

\[ y' = \left( \frac{u'}{u} - \frac{v'}{v} \right) \frac{u}{v} \]

\[ = \frac{(u'v - v'u)}{uv} \]

\[ = \frac{u'v - v'u}{v^2} \]
Example 7, p. 233 textbook. If time permits differentiate

\[ y = \frac{x^{3/4} \sqrt{x^2 + 1}}{(3x + 2)^5} \]

one or two ways.

\[ \ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2) \]

\[ \frac{y'}{y} = \frac{3}{4} \cdot \frac{1}{x} + \frac{2x}{2(x^2 + 1)} - \frac{15}{3x + 2} \]

\[ y' = \left( \frac{3}{4} \cdot \frac{1}{x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2} \right)^{3/4} \frac{\sqrt{x^2 + 1}}{(3x + 2)^5} \]

exercise: compute \( y' \) directly
\[ \frac{d}{dx} x^r = r x^{r-1} \]

\[ y = x^r \]

\[ \ln y = r \ln x \]

\[ \frac{y'}{y} = r \frac{1}{x} \]

\[ y' = r y \cdot \frac{1}{x} = r x^{r-1} \cdot \frac{1}{x} = r x^{r-1} \]

\[ \#55 \]

\[ s(t) = 2 \cos t + 3 \sin t \]

\[ s'(t) = -2 \sin t + 3 \cos t \]

\[ 2 \cos t + 3 \sin t = 0 \]
to find when the ball passes through equilibrium we need to solve

\[ s(t) = 2 \cos t + 3 \sin t = 0 \]

\[ 2 \cos t = -3 \sin t \]

\[ \frac{2}{3} = -\tan t \]

\[ \tan t = -\frac{2}{3} \]

\[ t = \arctan \left( -\frac{2}{3} \right) \]

but we want + positive

\[ t = \arctan \left( -\frac{2}{3} \right) + \pi \]

+ rounded to nearest \( \frac{1}{10} \), \( t \approx 2.55 \)

To compute the max displacement, we compute \( t \) such that the derivative is zero

\[ s' = -2 \sin t + 3 \cos t = 0 \]
\[ 3 \cos t = 2 \sin t \]

\[ \frac{3}{2} = \tan t \]

\[ t = \arctan \frac{3}{2} \]

To find the actual displacement we evaluate \( s(t) \) at \( \arctan \frac{3}{2} \).

Rounded to the nearest \( \frac{1}{100} \), this gives 3.61.
# 10 HW 6

\[ f(x) = \frac{x}{x+1} \]

\[ f'(x) = \frac{x+1 - x}{(x+1)^2} = \frac{1}{(x+1)^2} \]

\[ z = mx + b \]

\[ 2 - \frac{x}{x+1} \]

\[ \frac{2(x+1)^2 - x(x+1)}{1-x} = 1 \]

\[ 2(x^2 + 2x + 1) - x^2 - x = 1 - x \]
\[ x^2 + 4x + 1 = 0 \]

\[ x = \frac{-4 \pm \sqrt{16 - 4 \cdot 1}}{2} \]

\[ = \frac{-4 \pm \sqrt{12}}{2} \]

\[ = -2 \pm \sqrt{3} \]

---

Completing the square:

\[ x^2 + 4x + 1 = 0 \]

\[ 1 + 3 \]

\[ x^2 + 4x + 9 = 9 \]

\[ (x + 2)^2 = 3 \]

\[ x + 2 = \pm \sqrt{3} \]

\[ x = -2 \pm \sqrt{3} \]

\[ x^2 + 4x + 2^2 + 1 = 4 \]
\[ \frac{d}{dx} \left( 4x^2 \right) = 9x^4 \quad f'(x) \]

\[ f'(4x^2) \cdot 8x = 9x^4 \]

\[ f'(4x^2) = \frac{9}{8} x^2 \]

\[ f'(z) = \frac{9}{64} \left( 4x^2 \right)^{3/2} \]

\[ \frac{9}{64} \left( 4x^2 \right)^{3/2} = \frac{9}{8} x^3 \]

\[ C \left( 4x^2 \right)^{3/2} = \frac{9}{8} x^3 \]

\[ C = \frac{9}{64} \]
\[ d(t) = \sqrt{w^2 t^2 + v^2 (t-1)^2} \]
\[ = \left( w^2 t^2 + v^2 (t-1)^2 \right)^{1/2} \]
\[ d' = \frac{1}{2} \left( w^2 t^2 + v^2 (t-1)^2 \right)^{-1/2} \left( 2w^2 t + 2(t-1)v \right) \]
\( (x, y) = (3 \sin t, 3 \cos t) \)

\[ y(x) = \sqrt{9 - x^2} \]

\[ y'(x) = \frac{1}{2} \frac{-2x}{\sqrt{9 - x^2}} = \frac{-x}{\sqrt{9 - x^2}} \]

\[ \frac{x}{y} = \frac{9}{x - 15} \]

\[ x^2 + y^2 = 9 \]

\[ -x(x - 15) = y^2 \]

\[ x(15 - x) = y^2 \]

\[ x^2 + 15x - x^2 = 9 \]

\[ 15x = 9 \]

\[ x = \frac{9}{15} = \frac{3}{5} \]
\[
\frac{2}{5} = 3 \sin t
\]

\[
t = \arcsin \frac{1}{5} \text{ rad.}
\]

\[
= \frac{180 \arcsin \frac{1}{5}}{\pi} \text{ deg}
\]
Geometric

\[ \sin t = \frac{3}{15} \]

\[ t = \arcsin \frac{3}{15} \]