Math 1310-4
Notes of 9/20/19

• Continue with Differentiation Rules. So far we have:

• throughout: $c$ is constant, $f$ and $g$ are functions of $x$, a prime denotes differentiation with respect to $x$.

• The constant factor rule. You can multiply with a constant factor $c$ before or after differentiation.

$$ (cf)' = cf' = c(f') $$

• The Power Rule

$$ \frac{d}{dx} x^n = nx^{n-1}, \quad n = 1, 2, \ldots $$

• The sum rule

The derivative of the sum is the sum of the derivatives

$$ (f + g)' = f' + g' $$

• The difference rule

The derivative of the difference is the difference of the derivatives

$$ (f - g)' = f' - g' $$
• The Product Rule

The derivative of the product does not equal the product of the derivatives.

\[(fg)' \neq f'g' \]

(The danger symbol is from Knuth’s TeX book.)

Instead:

\[ (fg)' = f'g + fg' \]

• The Quotient Rule

the derivative of the quotient does not equal the quotient of the derivatives.

\[ \left( \frac{f}{g} \right)' \neq \frac{f'}{g'} \]

Instead

\[ \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \]

• The exponential equals its own derivative

\[
\frac{d}{dx} e^x = e^x \quad \text{and} \quad \frac{d}{dx} a^x = \ln a \times a^x
\]
• let’s do examples.

Suppose $n$ is a positive integer, $n = 1, 2, 3\ldots$

$$
\frac{d}{dx} x^{-n} = \frac{d}{dx} \frac{1}{x^n} = \frac{0 - n x^{n-1}}{(x^n)^2} = \frac{-n x^{n-1}}{x^{2n}} = \frac{-n}{x^{2n-(n-1)}} = \frac{-n}{x^{1+n}} = -n x^{-n-1}
$$

$$
\left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}
$$

$$
\frac{u^M}{u^N} = \frac{u^{M-N}}{u^N} = \frac{1}{u^{N-M}}
$$

$$
\frac{d}{dx} x^{-n} = -n x^{-n-1}
$$
\[ f'(x) = \frac{-8 \cdot 2x}{(x^2 + 4)^2} \]

Differentiate and graph

\[ f(x) = \frac{8}{x^2 + 4} \]

Note: This is a special case of a curve called the **Witch of Agnesi**. (Google it if you want to learn more.)

\[ f''(x) = \frac{-16(x^2 + 4)^2 + 16x(x^2 + 4)}{(x^2 + 4)^3} \cdot 4x \]

\[ = \frac{-16(x^2 + 4) + 64x^2}{(x^2 + 4)^3} \]

\[ = \frac{48x^2 - 64}{(x^2 + 4)^3} \]

\[ = \frac{16(3x^2 - 4)}{(x^2 + 4)^3} \]

\[ x = \pm \sqrt[3]{\frac{4}{3}} \]

\[ 3x^2 - 4 = 0 \]

\[ x^2 = \frac{4}{3} \]

\[ x = \pm \sqrt[3]{\frac{4}{3}} \]
Figure 1. The Witch of Agnesi.
Figure 2. More Derivatives.

Major Lesson: Differentiation simplifies polynomials, but complicate rational functions!

exercise: label the graphs
Alternative Forms of the Quotient Rule

• The quotient rule simplifies if the numerator is constant. We get

\[
\left( \frac{1}{g} \right)' = \frac{1'g - 1 \times g'}{g^2} = -\frac{g'}{g^2}
\]

since the derivative of 1 is zero.

• We can think of the quotient \( \frac{f}{g} \) as the product \( f \times \frac{1}{g} \) and differentiate the quotient by applying the product rule to the equivalent product:

\[
\left( \frac{f}{g} \right)' = \left( f \times \frac{1}{g} \right)' = f' \times \frac{1}{g} - f \times \frac{g'}{g^2}
\]

We get the same answer of course because by writing the result over the common denominator we get

\[
\frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}.
\]

• It’s always useful to think of a concept in many different ways.
3.3 Derivatives of Trigonometric Functions

- The trigonometric functions:

\[ \tan \theta = \frac{\sin \theta}{\cos \theta}. \]

- Sometimes the reciprocals of these functions are also used:

\[ \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta} \quad \text{and} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}. \]

- These are called secant, cosecant, and cotangent, respectively, but we will usually only use sine, cosine, and tangent.

\[ \text{Figure 3. Definition of sin and cos.} \]
Can we guess the derivative of the sin curve?

Figure 4. \( y = \sin x \).

\[
\frac{d}{dx} \sin x = \cos x
\]

Here is a computer generated image of the derivative:

Figure 5. \( y = \sin x \) and its derivative.
• Sure enough, 
\[
\frac{d}{dx} \sin x = \cos x \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x.
\]

• We’ll discuss why tomorrow, but first some differentiations:

\[
\frac{d}{dx} e^x \sin x = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)
\]

\[
\frac{d^2}{dx^2} e^x \sin x = e^x (\sin x + \cos x) + e^x (\cos x - \sin x) = 2e^x \cos x
\]

\[
\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}
\]
HW 5 # 16

\[ f(x) = ax^3 + bx^2 + cx + d \]

horiz. tangents at \((-2, 4)\) \((2, -2)\)

\[ f'(x) = 3ax^2 + 2bx + c \]

\[ f'(-2) = 12a - 4b + c = 0 \quad ? \quad b = 0 \]

\[ f'(2) = 12a + 4b + c = 0 \quad \text{?} \quad b = 0 \]

\[ f(-2) = -8a + 4b - 2c + d = 4 \]
\[ f(2) = 8a + 4b + 2c + d = -2 \]

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HW 5 # 18

\[ y = x + 6 \]

\[ m = \frac{1}{2} \]

\[ y = \frac{1}{2} + b \]

\[ b = -\frac{1}{4} \]

\[ \frac{1}{4} = y = \frac{1}{2} + b = \frac{1}{4} \]

\[ x = \frac{1}{2} \]

\[ y = \frac{1}{4} \]

\[ y' = 2x \geq 1 \]
\[
\begin{align*}
-\frac{m}{m} & = \frac{1}{m} \\
-\frac{1}{m} & = m \\
\end{align*}
\]
\[ h(t) = -\frac{1}{2} gt^2 + Vt + H \quad \text{H initial height} \]
\[ V \text{ initial velocity} \]

\[ V(t): h'(t) = -gt + V \]
\[ h''(t) = -g \]

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**How long will it stay in the air?**

\[ h(t) = -\frac{1}{2} gt^2 + Vt + H = 0 \]

*quadratic eqn, solve it*

\[ v(t) = -gt + V = 0 \]

\[ t = \frac{V}{g} \quad \text{time of max height} \]

\[ \text{max height} = h \left( \frac{V}{g} \right) \]

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\[ V \text{ unknown velocity of the arrow} \]
\[ V^2 = V_v^2 + V_h^2 = 2V_v^2 \]

\[ h(t) = -\frac{1}{2} gt^2 + V_v t = 0 \quad t = 0 \]

\[ t = \frac{V_v}{g} \]

\[ d = \frac{V_h \cdot t}{2} = \frac{2V_v V_h}{g} = \frac{2V_v^2}{g} = \frac{V^2}{g} \]

\[ V = \sqrt{\frac{dg}{400}} \quad 100 \text{ ft/s} \]

\[ 400 \quad 32 \text{ ft/s}^2 \]