Reminders

- Tomorrow: Review.
- Read the online notes before class.
- Tomorrow I will answer questions.

Derivatives of Exponentials

- Can we guess the derivative of $f(x) = e^x$?
• First the Facts!

\[ \frac{d}{dx} e^x = e^x \]

and, more generally

\[ \frac{d}{dx} a^x = \ln a \times a^x. \]

• The exponential equals its own derivative. Any exponential and its derivative are proportional.

• This makes sense in terms of the graph of the exponential and its tangent \( y = x + 1 \) at the point (0,1).

![Graph](image)

**Figure 1.** The exponential and its tangent \( y = x + 1 \).

• That’s the essence of exponential growth. The rate of change (the slope of the tangent) is proportional to what’s growing.
\[ \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

- why?
- First note that
\[
\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} \\
= ka^x \quad \text{where} \quad k = \lim_{h \to 0} \frac{a^h - 1}{h}. \tag{1}
\]

- This means that any exponential is proportional to itself!
- Of course we have to be sure that \( \lim_{h \to 0} \frac{a^h - 1}{h} \) exists. That’s a little subtle, but the limit does exist.
- Moreover, we defined (on page 57 of the textbook) the number \( e \) to be that number for which
\[
\lim_{h \to 0} \frac{e^h - 1}{h} = 1.
\]
- So we get from equation (1) that indeed
\[
\frac{d}{dx} e^x = e^x.
\]
- Recall that \( e \) is a real number that can be computed to a high degree of accuracy\(^{-1}\). It’s decimal expansion starts out as
\[
e = 2.71828182845904523536028747135266249775724709369995957496696676\ldots
\]
- \( e \) is irrational, like \( \pi \). Actually, both \( e \) and \( \pi \) are transcendental, which means they are not roots of any polynomial with rational coefficients.

\(^{-1}\) According to the wikipedia, \( e \) has been computed to “trillions of digits”. In Maple try the statement `Digits:=N; evalf(exp(1))`; where \( N \) is your favorite (reasonable) positive integer.
• So what about the constant of proportionality

\[ k = \lim_{h \to 0} \frac{a^h - 1}{h} \]

in

\[ \frac{d}{dx} a^x = ka^x? \]

• That’s a little trickier:

\[
\begin{align*}
  k &= \lim_{h \to 0} \frac{a^h - 1}{h} \\
  &= \lim_{h \to 0} \frac{(e^{\ln a})^h - 1}{h} \\
  &= \lim_{h \to 0} \frac{e^{h \ln a} - 1}{h} \\
  &= \lim_{h \to 0} \left( \frac{e^{h \ln a} - 1}{h \ln a} \right) \ln a \\
  &= \lim_{h \to 0} \left( \frac{e^z - 1}{z} \right) \times \ln a \\
  &= \ln a \\
\end{align*}
\]

setting \( h \ln a = z \)
The Product Rule

• The derivative of the sum is the sum of the derivatives.

• The derivative of the product does not equal the product of the derivatives!

• Example: \[ \frac{d}{dx} x^2 = \frac{d}{dx}(x \cdot x) = 1 \cdot x + x \cdot 1 = 2x \]

\[ x^2 = x \times x \quad \text{but} \quad \frac{d}{dx} x^2 = 2x \neq 1 \times 1 = \frac{d}{dx} x \times \frac{d}{dx} x. \]

• Instead, we have the Product Rule:

\[ (fg)' = f'g + fg' \]

or, more elaborately,

\[ \frac{d}{dx} [f(x)g(x)] = \frac{d}{dx} [f(x)]g(x) + f(x) \frac{d}{dx} [g(x)] \]

• Why?

\[ \frac{d}{dx} (f(x)g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \]

\[ = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x)}{h} \]

\[ \frac{3(4+5)}{7} = 3 \cdot \frac{4+5}{7} \]
An easy way to remember the product rule is to note that it has two terms, each of which is obtained by considering one of the factors constant:

\[(uv)' = u'v + uv'.\]
• Examples:

\[ f(x) = x^2 = x \times x \]

\[ f(x) = x + x = 2x \]

\[ f(x) = xe^x. \]

\[ \frac{d}{dx} xe^x = e^x + xe^x \]

\[ \frac{d}{dx} x^2 e^x = 2xe^x + x^2 e^x \]
An alternative Derivation of the Power Rule

\[ \frac{d}{dx} x^n = n \cdot x^{n-1} \]

\[ n = 1 \]
\[ n = 2 \]

\[ \frac{d}{dx} x^n = \frac{d}{dx} x \cdot x^{n-1} \]
\[ = x^{n-1} + x \cdot (n-1) x^{n-2} \]
\[ = x^{n-1} + (n-1) x^{n-1} \]
\[ = (1 + n-1) x^{n-1} \]
\[ = n \cdot x^{n-1} \]
The Quotient Rule

\[
\left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}
\]

or, more elaborately,

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\frac{d}{dx} [f(x)]g(x) - f(x)\frac{d}{dx} [g(x)]}{g^2(x)}
\]

• Why?

\[
\frac{d}{dx} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}
\]

\[
= \lim_{h \to 0} \frac{g(x)f(x+h) - f(x)g(x+h)}{h \ g(x) \ g(x+h)}
\]

\[
= \lim_{h \to 0} \frac{\ g(x) \ f(x+h) - f(x) \ g(x) + f(x)g(x) - f(x)g(x+h)}{h \ g(x) \ g(x+h)}
\]

\[
= \lim_{h \to 0} \frac{g(x)f(x+h) - g(x)f(x)}{g(x)} = \frac{f'g - fg'}{g^2}(x)
\]
• The quotient rule

\[ \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2} \]

is not as intuitive as the product rule. It’s the closest thing to something we need to memorize. On the other hand, in due course we will have used it so many times that we won’t be able to forget it!
\[
\frac{f'g - fg'}{g^2}
\]

- **Examples:**

\[
f(x) = \frac{x - 1}{x - 2}
\]

\[
f'(x) = \frac{(x-2) - (x-1)}{(x-2)^2} = \frac{-1}{(x-2)^2}
\]
\[
\frac{u'v - uv'}{v^2} \quad \quad \left( \frac{N}{D} \right)' = \frac{N'D - ND'}{D^2}
\]

\[f(x) = \frac{e^x}{x}\]

\[f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}\]
\[ f(x) = \frac{x^2 + 1}{x + 1} \]

\[
\begin{align*}
\frac{f(x)}{x} &= \frac{2x(x + 1) - (x^2 + 1)}{(x + 1)^2} \\
&= \frac{x^2 + 2x - 1}{(x + 1)^2}
\end{align*}
\]