Announcements

- Reminder: Today, and every Monday, I will stay after class for the next session (until 12:40) and answer questions from anybody who would like to stay.

- today: Start with chapter 3.

- tomorrow: More on Derivatives

- Wednesday, 9/18: review. The notes are now online. Read them before class. I will answer questions during class.

- Thursday, 9/19: The lab assistants will give their own review.

- Friday, 9/20: Exam 1, on Chapter 2.

- There is only one way to prepare for an exam: Make sure you understand the subject.

- Exam 2 (on 10/18) will be all on computing derivatives, i.e., applying differentiation rules.

- Differentiation rules form the contents of chapter 3.
• **Constant Multiple Rule:** (page 177)

\[
\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)
\]

This follows from the limit property

\[
\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)
\]

which we discussed on Friday.

• **The Sum and Difference rules** (pages 177 and 178)

\[
\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

and

\[
\frac{d}{dx} (f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)
\]

follow from the limit properties

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

\[
\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
\]

which we also discussed on Friday. It’s useful, and a good exercise, to think about how the limit properties imply the differentiation formulas. Details are in the textbook. We’ll skip them, but do ask me questions if you have any!

\[\text{Ex.} \quad f(x) = |x - 7|\]
\[g(x) = -f(x) + x\]
\[f(x) + g(x) = x\]
The Power Rule

The power rule asserts that

\[ \frac{d}{dx} x^r = r x^{r-1} \]

for all real exponents \( r \). So far we have verified this directly for \( r = 0, 1, 2, 3, -1, \frac{1}{2} \).

It will take several steps, and several meetings, to verify the rule for all exponents.

We begin by assuming that the exponent is a positive integer. In this context it is usually denoted by \( n \). Thus we want to show that

\[ \frac{d}{dx} x^n = nx^{n-1}, \quad n = 1, 2, 3, \ldots \]

We already saw on Friday that this can be done by using the Binomial Theorem. That theorem, however, is not as well understood and appreciated as it should be, so let’s review.

\[ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k} \]

\[ = x^n + nx^{n-1}y + \frac{n(n-1)}{2} x^{n-2} y^2 + \ldots + \binom{n}{k} x^n y^{n-k} + \ldots + y^n. \]

where

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

and

\[ n! = \begin{cases} 1 & \text{if } n = 0 \\ 1 \times 2 \times 3 \ldots \times n & \text{i } n > 0 \end{cases} \]
$n!$ is pronounced \textit{n-factorial}. A small table of the first few factorials is

\begin{center}
\begin{tabular}{ccccccccccc}
$n$ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
n! & 1 & 1 & 2 & 6 & 24 & 120 & 720 & 5040 & 40,320 & 362,880 & 3,626,800 \\
\end{tabular}
\end{center}

\(\binom{n}{k}\) is pronounced \textit{n choose k}. This is because \textit{n choose k} is the number of ways in which you can choose \(k\) items from a set of \(n\) items. For example, there are

\[
\binom{130}{5} = \frac{130!}{5! \times 125!} = 286,243,776
\]

ways to choose a basketball team of 5 players from our class of 50 students.

For the first few values of \(n\) the binomial theorem says:

\begin{align*}
n = 0: & \quad (x + y)^0 = 1 \\
n = 1: & \quad (x + y)^1 = x + y \\
n = 2: & \quad (x + y)^2 = x^2 + 2xy + y^2 \\
n = 3: & \quad (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \\
n = 4: & \quad (x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
n = 5: & \quad (x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\
\end{align*}

A useful way to think of \((x + y)^n\) is as the sum of all products that you can form by picking exactly one term from each of the factors in

\[
(x + y)^n = (x + y) \times (x + y) \times \ldots \times (x + y).
\]

\(n\) factors \((x + y)\)

There are \(\binom{n}{k}\) ways to pick \(k\) \(x\)’s and \((n - k)\) \(y\)’s which explains the coefficient \(n\) choose \(k\).
The binomial coefficients \( \binom{n}{k} \) form Pascal’s Triangle:

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\ldots
\end{array}
\]

Each entry is the sum of the two entries above it.

You can see why this is so by considering how the \( n = 4 \) row, for example, is obtained from the \( n = 3 \) row.

\[
(x + y)^4 = (x + y)^3 \times (x + y)
\]

\[
= (x^3 + 3x^2y + 3xy^2 + y^3) x \\
+ (x^3 + 3x^2y + 3xy^2 + y^3) y \\
= x^4 + 3x^3y + 3x^2y^2 + xy^3 \\
+ x^3y + 3x^2y^2 + 3xy^3 + y^4 \\
= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\]

- As an exercise you may want to prove the binomial theorem by induction.
Having built all this machinery we will really use only the first two terms in the binomial formula . . .

Suppose \( f(x) = x^n \). Then

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}
\]

\[
= \lim_{h \to 0} \frac{x^n + nh x^{n-1} + h^2 \left[ \binom{n}{2} x^{n-2} + \binom{n}{3} hx^{n-3} + \ldots \right] - x^n}{h}
\]

\[
= \lim_{h \to 0} n x^{n-1} + h \left[ \binom{n}{2} x^{n-2} + \binom{n}{3} hx^{n-3} + \ldots \right]
\]

\[
= nx^{n-1}.
\]

As we discussed on Friday, we can now compute derivatives of any polynomial.

\[
\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \ldots - x^n}{h}
\]

\[
= nx^{n-1}.
\]
• **Example:**

Compute all derivatives of

\[ p(x) = x^5 - x^4 - 2x^2 + 3x + 17. \]

\[
\begin{align*}
p'(x) &= 5x^4 - 4x^3 - 4x + 3 \\
p''(x) &= 20x^3 - 12x^2 - 4 \\
p'''(x) &= 60x^2 - 24x \\
p''''(x) &= 120x - 24 \\
p'''''(x) &= 120 \\
p''''''(x) &= 0
\end{align*}
\]

\[
\frac{d^{17}}{dx^{17}} x^{17} = 17!
\]

• **(Important) Lessons:**

• The derivative of a polynomial is a polynomial.

• Differentiation reduces the degree of a polynomial by 1.

• The \( n \)-th derivative of a polynomial of degree \( n \) is constant.

• In fact,

\[
\frac{d^n}{dx^n} x^n = n!
\]

• If the degree of the derivative exceeds the degree of a the polynomial then the derivative is zero.
• Example:

What is the derivative of

\[ p(x) = (x + 1)^2 \]?

\[
\frac{d}{dx} (x+1)^2 = \frac{d}{dx} (x^2 + 2x + 1)
\]

\[
= 2x + 2
\]

\[
= 2(x+1)
\]
• Example:

What is the derivative of

\[ p(x) = (x^2 + 1)^2? \]

\[
\frac{d}{dx} (x^2 + 1)^2 = \frac{d}{dx} (x^4 + 2x^2 + 1)
\]

\[
= 4x^3 + 4x
\]

\[
= 4x(x^2 + 1)
\]

\[
= 2 \cdot 2x(x^2 + 1)
\]
What about the derivatives of \( p(x) = (x+1)^{100} \) and \( q(x) = (x^2 + 1)^{100} \). 

\[
\frac{d}{dx} (x+1)^{100} = \frac{d}{dx} \sum_{k=0}^{100} \binom{100}{k} x^{100-k} \\
= \sum_{k=0}^{100} \binom{100}{k} x^{k-1} \\
= \sum_{k=1}^{100} \binom{100}{k-1} x^{k-1} \\
= 100 (1+x)^{99}
\]
\[
\frac{d}{dx} \left( 1 + x^2 \right)^{100} = \frac{d}{dx} \sum_{k=0}^{100} \binom{100}{k} x^{2k} = 2x \cdot 100 \left( 1 + x^2 \right)^{99}
\]

\[
\int_0^1 f(x) g(x) \, dx + \int f(x) g(x) \, dx = f(x) g(x) \Big|_0^1 2
\]

\[
\frac{d}{dx} x^2 = \frac{d}{dx} x \cdot x = 2x \neq 1 \cdot 1
\]
\[
\lim_{x \to 0} \frac{f(x)}{x} = \frac{f(a)}{a} \quad \text{if} \quad a = 0
\]

\[
\int f(x) = 7x
\]

\[
(x-1)^2 + y^2 = 1
\]
\[
\begin{align*}
    m &= \frac{Y_Q - Y_P}{X_Q - X_P} = \frac{Y_R - Y_P}{X_R - X_P} \\
    \frac{Y_Q - Y_P}{X_Q} &= \frac{-Y_P}{X_R} \\
    -\frac{X_R}{Y_P} &= \frac{X_Q}{Y_Q - Y_P} \\
    X_R &= \frac{-Y_P X_Q}{Y_Q - Y_P} \\
    &= \frac{\tau X_Q}{Y_P - Y_Q}
\end{align*}
\]

\[
\begin{align*}
    (X_Q - 1)^2 + Y_Q^2 &= 1 \\
    X_Q^2 + Y_Q^2 &= Y_Q^2 \\
    Y_Q &= \sqrt{r^2 - X_Q^2} \\
    (X_Q - 1)^2 + r^2 - X_Q^2 &= 1
\end{align*}
\]
\[
X_Q^2 - 2x_Q + 1 + r^2 - x_Q^2 = 1
\]
\[- 2x_Q + r^2 = 0
\]
\[x_Q = \frac{r^2}{2}
\]
\[y_Q^2 = r^2 - x_Q^2 = r^2 - \frac{r^4}{4}
\]
\[y_Q = \sqrt{r^2 - \frac{r^4}{4}}
\]
\[X_R = \frac{r \sqrt{y_Q}}{\gamma_p - y_Q}
\]
\[= r - \frac{r^2}{2}
\]
\[= \sqrt{r^2 - \frac{r^4}{4}}
\]
\[= \frac{r^2}{2} \left( 1 - \sqrt{1 - \frac{r^2}{4}} \right)
\]
\[= \frac{\sqrt{2}^2}{2} \left( 1 - \sqrt{1 - \frac{r^2}{4}} \right)
\]
\[= \frac{y_Q^2}{2} \left( 1 + \sqrt{1 - \frac{r^2}{4}} \right)
\]
\[= \frac{y_Q^2}{2} \left( 1 - \left( 1 - \frac{r^2}{4} \right) \right)
\]
\[(a + b)(a - b) = a^2 - b^2\]
\[
\begin{align*}
&= \frac{r^2}{2} \left( 1 + \sqrt{1 - \frac{r^2}{4}} \right) \\
&= 2 \left( 1 + \sqrt{1 - \frac{r^2}{4}} \right) \\
\lim_{r \to 0} \left( \right) &= 4 \\
\end{align*}
\]

\[
\begin{align*}
\hat{r} &= 1.67 \\
f'(r) &= -\frac{2}{r^2} \\
f(1.67) &= 372076.60 \\
f(1.66) &= 372504 \\
\end{align*}
\]

\[
\begin{align*}
f(1.671) &= 372031.44 \quad \text{0.0428} \\
-4.28 \\
\end{align*}
\]
\[ f(x) = -3x + c \]

\[ Q = \frac{f(x+h) - f(x)}{h} = Ah + Bx + D \]

\[ h = \frac{-3(x+h) + c - (-3x + c)}{h} \]

\[ h = -3 \]

\[ \#15 \quad L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \]

\[ \lim_{v \to c} L_0 \sqrt{1 - \frac{v^2}{c^2}} = \]
\[
\lim_{v \to c} \frac{v^2}{c^2} = 1
\]

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[
f(x+h) - f(x) = -5h^2 + 4hx - 5hx^2 - 2h^2 + 2h^3
\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \left( -5h^2 + 4hx - 5hx^2 - 2h^2 + 2h^3 \right)
\]

\[
= -5x^2 + 4x
\]

\[
f'(x) = -\frac{5}{3}x^3 + 2x^2 + \xi
\]