2.6 Derivatives and Rates of Change

- We are finally entering the heart of Calculus.
- Recall our discussions of velocity and slopes of tangents.
- If $s(x)$ is location at time $x$, then the **average velocity** in the time interval $[a, a + h]$ is

$$v_{\text{avg}} = \frac{\text{displacement}}{\text{time}} = \frac{s(a + h) - s(a)}{h}$$

(1)

- The **instantaneous velocity** is

$$v(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$$

(2)

![Figure 1. Average and Instantaneous Velocity.](image)
• Geometrically, average and instantaneous velocity are the slopes of the secant and tangent lines, respectively.

It’s crucial to understand this idea.
More notation. The Greek Letter $\Delta$ (capital delta) is often used to denote differences. Letting

$$\Delta x = (a + h) - a = h \quad \text{and} \quad \Delta y = f(a + h) - f(a)$$

equation (1) turns into

$$v_{\text{avg}} = \frac{\Delta y}{\Delta x} \quad (3)$$

and (2) turns into

$$v(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad (4)$$
• We are now ready for one of two central definitions of Calculus:

• **Definition 4 (page 138):** The derivative of a function \( f \) at a number \( a \), denoted by \( f'(a) \), is

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

if this limit exists.

Another form of this definition is obtained by writing

\[
h = x - a.
\]

This gives the equivalent Definition 5.

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]

• \( f' \) is pronounced f-prime.

• \( f' \) is the derivative of \( f \).

• The generalization of the concept of “velocity” is **rate of change**. As for velocity we can define the **average rate of change** and **instantaneous rate of change**.

• The derivative tells us the rate of change of the function.

• Velocity is the derivative of location.

• Acceleration is the derivative of velocity.

• The derivative of acceleration is sometimes called “jerk”.

• Of course, the independent variable could be something other than time. For example, pressure or temperature could depend on location.
Let’s do a specific example. Ignoring air resistance, an object falling from an initial height \( h_0 \) with an initial velocity \( v_0 \) and experiencing an acceleration of \(-g\) feet per second squared (near the surface of the earth \( g = 32 \)) is at a height

\[
h(t) = -\frac{g}{2}t^2 + v_0 t + h_0
\]

after \( t \) seconds. Compute its velocity \( v(t) = h'(t) \). (The textbook uses \( v(a) \) but \( v(t) \) is more practical and more common.)

\[
h'(t) = \lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{-\frac{g}{2} (t + \Delta t)^2 + v_0 (t + \Delta t) + h_0 - \left(\frac{g}{2} t^2 + v_0 t + h_0\right)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{-\frac{g}{2} \left((t + \Delta t)^2 - t^2\right) + v_0 (t + \Delta t - t)}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{-\frac{g}{2} (t \Delta t + (\Delta t)^2) + v_0 \Delta t}{\Delta t}
\]

\[
(t + \Delta t)^2 = t^2 + 2t \Delta t + (\Delta t)^2
\]

\[
= \lim_{\Delta t \to 0} \frac{-\frac{g}{2} \left(2t \Delta t + (\Delta t)^2\right) + v_0 \Delta t}{\Delta t}
\]
\[
\lim_{{\Delta t \to 0}} \left[-\frac{g}{2} (2t + \Delta t) + v_0 \right] = -gt + v_0 \\
\therefore v(t) = h'(t)
\]
Figure 2. $h_0 = 2$, $v_0 = 0$, $g = 2$, $h(t) = 2 - t^2$, $h'(t) = -2t$. 
• Another example: Suppose

\[ f(x) = x^3. \]

Compute the equation of the tangent at the point \((1, 1)\).

\[
\frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}
\]

\[
= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}
\]

\[
= \lim_{h \to 0} \left(3x^2 + 3hx + h^2\right)
\]

\[
= 3x^2
\]
Figure 3. $f(x) = x^3$, $f'(x) = 3x^2$. 
Given \( f(x) = \frac{1}{x^2 + 1} \)

compute \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2 + 1} - \frac{1}{x^2 + 1}}{h}
= \lim_{h \to 0} \frac{(x^2 + 1) - ((x+h)^2 + 1)}{h((x+h)^2 + 1)(x^2 + 1)}
= \lim_{h \to 0} \frac{-2hx - h^2}{h((x+h)^2 + 1)(x^2 + 1)}
= \lim_{h \to 0} \frac{-2x - h}{((x+h)^2 + 1)(x^2 + 1)}
\]

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\[\lim_{h \to 0} \frac{-2x - h}{(x+h)^2 + 1} (x^2 + 1) \]

\[= \frac{-2x}{(x^2 + 1)^2} \]

\[f(x) = \frac{1}{1 + x^2} = \gamma\]
Figure 4. \( f(x) = \frac{1}{x^2 + 1}; \) \( f'(x) = -2x/(x^2 + 1)^2.\)
A Table of Derivatives

So far this semester, we have computed the derivatives given in the following table. We use $x$ as the independent variable and also as the point at which we evaluate the derivative:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$f'(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0</td>
</tr>
<tr>
<td>$t$</td>
<td>1</td>
</tr>
<tr>
<td>$t^2$</td>
<td>$2t$</td>
</tr>
<tr>
<td>$t^3$</td>
<td>$3t^2$</td>
</tr>
<tr>
<td>$-\frac{g}{2}t^2 + v_0 t + h_0$</td>
<td>$-gt + v_0$</td>
</tr>
<tr>
<td>$\frac{1}{t}$</td>
<td>$-\frac{1}{t^2}$</td>
</tr>
<tr>
<td>$\sqrt{t}$</td>
<td>$-\frac{1}{2\sqrt{t}}$</td>
</tr>
</tbody>
</table>

- We will greatly expand this table, in fact within three weeks from now, just in time for the semester break, we will know how to differentiate any “elementary” function. Chapter 3 is dedicated to that discussion.