2.4 Continuity

Casually speaking, a function is continuous if its graph can be drawn without ever lifting the pencil. More precisely (p. 113, textbook):
A function \( f \) is continuous at a number \( a \) if

\[
f(a) = \lim_{x \to a} f(x).
\]  

(1)

\( f \) is continuous if it is continuous at all points in its domain.

Note that (1) says 3 things:
1. \( f(a) \) exists, i.e., \( a \) is in the domain of \( f \).
2. \( \lim_{x \to a} f(x) \) exists.
3. \( \lim_{x \to a} f(x) = f(a) \)

• Examples:

\( f(x) = x^2 + x + 1 \)

\( f(x) = \text{polynomial} \)

\( f(x) = \frac{1}{x^2} \)

\( f(x) = \frac{1}{x^2+1} \)

\( f(x) = \text{rational} \)
• More Examples

\[ f(x) = [x] \]

\[ f(x) = \sin \left( \frac{1}{x} \right) \text{ everywhere but } 0 \]

\[ f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & x \neq 0 \\ 0 & x = 0 \end{cases} \]

\[ f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \]

\[ \lim_{x \to 0} f(x) = f(0) = 0 \]

\[ \lim_{x \to 1.1} \left[ x \right] = 1 = [1.1] \]
One-sided Continuity

(p. 115) A function $f$ is **continuous from the right at a number** $a$ if

$$f(a) = \lim_{x \to a^+} f(x).$$  \hspace{1cm}(2)

Similarly for continuity from the left (exercise).

- **Example:**

  $$f(x) = [x].$$

$f$ is **continuous on an interval** if it is continuous at all points in that interval. If $f$ is defined on only one side of an endpoint we understand continuity at that endpoint meaning one-sided continuity from the appropriate side.
List of Continuous Functions

In the following list, the given functions are all continuous at all points in their domain

- polynomials.
- rational functions.
- power functions.
- Exponentials.
- rational functions
- trigonometric functions

Suppose $f$ and $g$ are continuous at a point $a$. Then the following are also continuous.

- $f + g$
- $f - g$
- $f \times g$
- $f/g$ so long as the denominator is non-zero.
- $f \circ g$ (A continuous function of a continuous function is continuous.)

Exercise: Think about $f^{-1}$. 
Intermediate Value Theorem

(page 120) Suppose that $f$ is continuous on the closed interval $[a, b]$, and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$. 
• Example:

Suppose \( f(x) = x^2 \), \([a, b] = [0, 1]\), and \( N = \frac{1}{2} \) Find \( c \).

\[
\begin{align*}
\quad f(0) &= 0 \\
\quad f(1) &= 1 \\
\quad f(x) &= x^2 = \frac{1}{2} \\
\quad x &= \frac{1}{\sqrt{2}} = c \\
\quad f(c) &= \frac{1}{2}
\end{align*}
\]

More generally: Suppose \( f(x) = x^n, \ n > 1, \ [a, b] = [0, 1] \), and \( N = \frac{1}{2} \) Find \( c \).

\[
\begin{align*}
\quad x^n &= \frac{1}{2} \\
\quad c &= x = \left(\frac{1}{2}\right)^{\frac{1}{n}} = \frac{1}{\sqrt[2^n]{2}}
\end{align*}
\]
• Example:

Does $e^x + x = 0$ have a solution?

$$f(x) = e^x + x \geq 0$$

$$f(0) = 1 > 0$$

$$f(1) = e + 1$$

$$f(-1) = \frac{1}{e} - 1 < 0$$
Method of Bisection.

Suppose $f(a)f(b) < 0$. Solve $f(x) = 0$.\[f\text{ continuous on } [a,b]\]
Return to $f(x) = e^x + x = 0$.

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Maple 2015 (X86 64 LINUX)

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| Type ? for help.

> restart:
> f := exp(x) + x:
> a := -1:
> b := 0:
> fa := subs(x = a, f):
> fb := subs(x = b, f):
>
> for i to 10 do
> c := (a + b)/2:
> fc := evalf(subs(x = c, f)):
> if fc < 0.0 then
> a := c:
> fa := fc:
> else
> b := c:
> fb := fc:
> end if:
> lprint(i, c, evalf(fc), a, b):
> end do:

1, -1/2, .1065306597, -1, -1/2
2, -3/4, -.2776334473, -3/4, -1/2
3, -5/8, -.897385715e-1, -5/8, -1/2
4, -9/16, .72828247e-2, -5/8, -9/16
5, -19/32, -.414975498e-1, -19/32, -9/16
6, -37/64, -.171758392e-1, -37/64, -1/2
7, -73/128, -.9637604e-2, -73/128, -9/16
8, -145/256, .11552020e-2, -73/128, -145/256
9, -291/512, -.19053596e-2, -291/512, -145/256
10, -581/1024, -.3753492e-3, -581/1024, -145/256
> lprint(evalf(a + b)/2):
- .5668945310
>
> quit:
```

memory used=0.8MB, alloc=8.3MB, time=0.05
Problem 35, page 122. For what value of the constant $c$ is the function $f$ continuous on $(\infty, \infty)$?

$$f(x) = \begin{cases} 
  cx^2 + 2x & \text{if } x < 2 \\
  x^3 - cx & \text{if } x \geq 2
\end{cases}$$

$x = 2$

$$cx^2 + 2x = 4c + 4$$

$$x^3 - cx = 8 - 2c$$

$$8 - 2c = 4c + 4$$

$$4 = 6c$$

$$c = \frac{2}{3}$$
Suppose $f(\theta)$ is the air pressure at the point with longitude $\theta$ on the equator. Can you find a continuous $2\pi$-periodic function $f$ so that no two diametrically opposed points on the equator have the same air pressure?

$$g(\theta) = f(\theta) - f(\theta + \pi)$$

$$g(0) = f(0) - f(\pi)$$

$$g(\pi) = f(\pi) - f(2\pi)$$

$$= f(\pi) - f(0)$$

$$= -g(0)$$
\[
\lim_{x \to 0} \frac{1}{x^2} = \infty
\]

\[
\frac{1}{x^2} > 10,000
\]

\[
x^2 < 10^{-20,000}
\]

\[
x < 10^{-10,000}
\]

\[
\lim_{x \to \infty} \frac{x^2 + 1}{x^2 + 2} = 1
\]

\[
\lim_{x \to \infty} \frac{3x^3 + 2x^2 + x + 1}{x^3 - 2x - 1} = 3
\]
\[ x^3 - 2x - 1 \]

\[ \begin{align*}
3x^3 + 2x^2 + x + 1 \\
3x^3 + 0x^2 - 6x - 3
\end{align*} \]

\[ 2x^2 + 7x + 4 \]

\[ R = 3 + \frac{2x^2 + 7x + 4}{x^3 - 2x - 1} \]

\[ \lim_{{x \to a}} \frac{x + \sqrt[3]{7}}{x + 2\sqrt[3]{7}} = \]

\[ \lim_{{x \to a}} \frac{x^{3/2} + x}{2x^{3/2} + x} \]
\[ \lim_{x \to a} \frac{f}{f} \]

\[ \lim_{x \to a} \left( 1 + \frac{x}{f} \right) = 1 \]

\[ = \lim_{x \to 00} \frac{x + 1}{f} = 1 \]
vertical asymptotes

when denominator = 0

x-axis as an asymptote.

rational function

cles denominator > deg numerator

horiz. non-zero asympt

deg numerator = deg denominator

slant:

deg numerator = deg denominator + 1

\[
\frac{x^2 + 3x + 1}{x + 1} = x + 2 - \frac{1}{x + 1}
\]
\[ \lim_{x \to 0} x^2 \cos(20\pi x) \]

\[-x^2 \leq x^2 \cos(20\pi x) \leq x^2\]