Math 1310-4 Notes of September 4, 2016

• We did not quite finish yesterday’s notes. Here is the last page from those notes:

Limits

• (Definition, page 95). We write

$$\lim_{x \to a} f(x) = L$$

and say “the limit of $f(x)$, as $x$ approaches $a$, equals $L$”, if we can make the values of $f(x)$ arbitrarily close to $L$ (as close to $L$ as we like) by taking $x$ to be sufficiently close to $a$ (on either side of $a$) but not equal to $a$.

• We saw

$$\lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} = 2a.$$ What are $f$, $x$, $L$, and $a$ in this context?

• We also saw today, for example:

$$\lim_{h \to 0} \frac{(a + h)^3 - a^3}{h} = 3a^2$$

$$\lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h} = \frac{1}{2\sqrt{a}}$$

$$\lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = -\frac{1}{a^2}$$

• We need to get more systematic about this kind of result.

• (We also saw today how important it is to understand algebra.)
2.2 Limits

• (Definition, page 95). We write

\[ \lim_{x \to a} f(x) = L \]

and say
- the limit of \( f(x) \), as \( x \) approaches \( a \), equals \( L \),
  if we can make the values of \( f(x) \) arbitrarily close to \( L \) (as close to \( L \) as we like) by taking \( x \) to be sufficiently close to \( a \) (on either side of \( a \)) but not equal to \( a \).

• This can also be written as

\[ f(x) \to L \quad \text{as} \quad x \to a \]

• Other ways of stating the fact verbally include:
  - \( f(x) \) approaches \( L \) as \( x \) approaches \( a \)
  - \( f(x) \) goes to \( L \) as \( x \) goes to \( a \).

• Some examples are obvious

\[ \lim_{x \to 0} \pi + x = \]

\[ \lim_{x \to 1} \pi + x = \]

\[ \lim_{x \to 1} \sqrt{3} + x = \]
• Here is a slightly trickier example:

\[
\lim_{x \to \pi} f(x) =
\]

where

\[
f(x) = \begin{cases} 
1 & \text{if } x = \pi \\
2 & \text{if } x \neq \pi 
\end{cases}
\]
• However, some limits aren’t obvious. Let’s look at two examples:

\[
\lim_{t \to 0} \frac{\sqrt{9+t^2} - 3}{t^2} = \quad \text{(Example 2, p. 97, textbook)}
\]

and

\[
\lim_{x \to 0} \frac{\sin x}{x} =
\]

• We will use 3 approaches to limit computation:
  – Evaluating at \( x \) close to \( a \), and guessing
  – Graphing
  – Algebraic

• Maple demonstration . . . you can use this maple code (without the line numbers), for example:

```plaintext
restart;
z:=(sqrt(t**2+9)-3)/(t**2);
evalf(subs(t=0.1,z));
evalf(subs(t=0.01,z));
evalf(subs(t=0.001,z));
evalf(subs(t=0.0001,z));
evalf(subs(t=0.00001,z));
plotsetup(ps,plotoutput='11a.ps',plotoptions='portrait, noborder,height=700,width=700');
plot(z,t=-1..1,numpoints=2000,thickness=2);
z:=(sin(x)/x);
evalf(subs(x=0.1,z));
evalf(subs(x=0.01,z));
evalf(subs(x=0.001,z));
evalf(subs(x=0.0001,z));
evalf(subs(x=0.00001,z));
plotsetup(ps,plotoutput='11b.ps',plotoptions='portrait, noborder,height=700,width=700');
plot(z,x=-1..1,numpoints=2000,thickness=2);
```

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This gives the following (redacted) output

```maple
> restart;

> z:=((sqrt(t**2+9)-3)/(t**2));

2 1/2
(t + 9) - 3
z := ---------------
    2
    t

> evalf(subs(t=0.1,z));

0.1666204000

> evalf(subs(t=0.01,z));

0.1666700000

> evalf(subs(t=0.001,z));

0.1670000000

> evalf(subs(t=0.0001,z));

0.2000000000

> evalf(subs(t=0.00001,z));

0.

> z:=(sin(x)/x);

z := ------
    x

> evalf(subs(x=0.1,z));

0.9983341665

> evalf(subs(x=0.01,z));

0.9999833334

> evalf(subs(x=0.001,z));

0.9999998333

> evalf(subs(x=0.0001,z));

0.9999999983

> evalf(subs(x=0.00001,z));

1.000000000
```

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• We also obtain these Figures:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{Graph of $\frac{\sqrt{9+t^2} - 3}{t^2}$.}
\end{figure}
and

Figure 2. Graph of \( \frac{\sin(x)}{x} \).
Algebraic calculation of $\lim_{t \to 0} \frac{\sqrt{9+t^2}-3}{t^2} =$
• Algebraic calculation of \( \lim_{x \to 0} \frac{\sin x}{x} = \)
• Example: What about the limit of $\sqrt{-x}$ as $x$ approaches zero?

• Recall our definition. We write

$$\lim_{x \to a} f(x) = L$$

and say

- **the limit of** $f(x)$, **as** $x$ **approaches** $a$, **equals** $L$,

  if we can make the values of $f(x)$ arbitrarily close to $L$ (as close to $L$ as we like) by taking $x$ to be sufficiently close to $a$ (on either side of $a$) but not equal to $a$. 
One-sided Limits

• Definition 2 (p.100), textbook:
  We write
  \[ \lim_{x \to a^-} = L \]
  and say the **left-hand limit of** \( f(x) \) **as** \( x \) **approaches** \( a \) **is equal to** \( L \) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) to be sufficiently close to \( a \) and \( x \) less than \( a \).

• Other ways of stating this fact include:

  – The **limit of** \( f(x) \) **as** \( x \) **approaches** \( a \) **from the left** is \( L \)
  – As \( x \) **approaches** \( a \) **from the left** \( f(x) \) **approaches** \( L \)
  – \( f(x) \) **goes to** \( L \) **as** \( x \) **approaches** \( a \) **from the left.**

• In the same way we can define a right-hand limit.

• **Exercise:** Modify the above language to define

  \[ \lim_{x \to a^+} = L \]

• Clearly:

  \[ \lim_{x \to a} f(x) = L \] if and only if \[ \lim_{x \to a^+} f(x) = L \] and \[ \lim_{x \to a^-} f(x) = L \]
The Heaviside Function

Figure 3. The Heaviside Function.

\[ H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \]

\[ \lim_{t \to 0^-} H(t) = \]

\[ \lim_{t \to 0^+} H(t) = \]
\[ \lim_{t \to 0} H(t) = \]
The limit may not exist

- What can go wrong?

\[
\lim_{x \to 0^-} f(x) =
\]

\[
\lim_{x \to 0^+} f(x) =
\]

\[
\lim_{x \to 0} f(x) =
\]

Figure 4. \( f(x) = \frac{1}{x^2} \).
The limit may not exist

Figure 5. $f(x) = 1/x$.

$$f(x) = \frac{1}{x}$$

$$\lim_{x \to 0^-} f(x) =$$

$$\lim_{x \to 0^+} f(x) =$$

$$\lim_{x \to 0} f(x) =$$
A Perverse Example

\[ f(x) = \sin \frac{1}{x} \]

Figure 6. \( f(x) = \sin \frac{1}{x}, -0.4 \leq x \leq 0.4 \).

\[ \lim_{x \to 0^-} \sin \frac{1}{x} = \]

\[ \lim_{x \to 0^+} \sin \frac{1}{x} = \]
Figure 7. $f(x) = \sin \frac{1}{x}, -0.04 \leq x \leq 0.04$.

$$\lim_{x \to 0} \sin \frac{1}{x} =$$
Figure 8. $f(x) = \sin \frac{1}{x}, -0.004 \leq x \leq 0.004.$
More Perverse Examples

\[ f(x) = \begin{cases} 
1 & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational}
\end{cases} \]

\[ f(x) = \begin{cases} 
x^2 & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational}
\end{cases} \]