Math 1220-3, 7, 90 — Fall 2020

Notes of 9/8/20

• Practice Exam was useful. We will have another practice run (on a smaller scale).

• Recall
  – An exponential function is of the form
    \[ f(x) = Ca^x. \quad a \neq 0 \quad a \neq 1 \]
  – It can also be written as
    \[ f(x) = Ce^{kx} = C a^x = C e^{\ln a^x} = C e^{(\ln a)\cdot x} \]
  – Query: What is \( k \)?
  – \( k \) is the rate constant, \( C = f(0) \) is the initial value, \( a \) is the base, \( x \) is the exponent. Very often, \( x \) is time. In that case we often use \( t \), instead of \( x \), as the independent variable.
  – An exponential function is proportional to its derivative.
  – Over a time interval of fixed length, an exponential function grows by a fixed percentage.
  – Equivalently, over a time interval of fixed length, an exponential function is multiplied by the same factor.

• Repetition implies importance!
Example 2, textbook: Suppose you are growing a culture of bacteria. You have 10,000 bacteria at noon and 40,000 at 2pm. How many will you have at 5pm? (Assume exponential growth.)

\[ 2^5 = 32 \]

\[ f(5) = 320,000 \]

\[
\begin{align*}
  f(x) &= c e^{kx} \quad c, k = ? \\
  f(0) &= 10,000 = c e^0 = c \\
  f(x) &= 10,000 e^{kx} \\
  f(2) &= 40,000 = 10,000 e^{2k} \\
  e^{2k} &= 4 \\
  2k &= \ln 4 \\
  k &= \frac{\ln 4}{2} \\
  f(x) &= 10,000 \cdot e^{\frac{\ln 4}{2} \cdot x} \\
  f(x) &= 10,000 \cdot 2^x
\end{align*}
\]
Radioactive Dating

- not everything grows.
- **exponential decay**: measured by **half life**, the time required to reduce a given amount or population by one half.
- Example: the half life of Carbon 14 ($^{14}C$) is 5,730 years.
- Try again
  \[ f(t) = e^{kt} \]
- We want $k$ to be such that
  \[ e^{5730k} = \frac{1}{2} \]
- This equation can be solved easily
  \[ k = \frac{\ln 1/2}{5730} \approx -0.000121. \]
- Problem 17 on p. 353 of the textbook describes the basis of radioactive dating. The procedure was worked out by Willard Libby in the 1940s. Libby later received the Nobel Prize for his work.
- Carbon 12, ($^{12}C$), is stable. It does not decay (on the time scales of interest). As stated
above, Carbon 14, \((^{14}C)\), does decay. However, in the atmosphere, it is also replenished by the interaction of cosmic rays with the Carbon 14 in the atmosphere. As a result, the ratio of

\[ R = \frac{^{14}C}{^{12}C} \]

in the atmosphere is essentially constant.

- Living Organisms interact with the atmosphere and that ratio in the organism’s body is the same as in the atmosphere.
- When the organism dies the interaction with the atmosphere stops, \(^{14}C\) in the organism continues to decay, and the ratio \(R\) decreases. By measuring the ratio we can estimate (quite accurately) the length of time that has passed since the organism died.
• Example: The ratio $R$ in the hair of a frozen mammoth found in the Siberian Tundra is 30% of its ordinary value. How long ago did the mammoth die?

\[
\text{\simeq 10,000 years}
\]

\[
e^{kt} = 0.3 \quad k = -0.000121
\]

\[
\ln e^{kt} = kt = \ln 0.3
\]

\[
t = \frac{\ln 0.3}{k} = 9950 \text{ years}
\]

• Many variations of this technique are available and allow scientists to determine the ages of many minerals and other material.
Newton’s Law of Cooling

- The rate at which an object cools or warms is proportional to the difference of the temperature of the object and the temperature of the surrounding medium. Let $T(t)$ denote the temperature of the object at time $t$ and $T_1$ the constant temperature of the surrounding medium.

- Then we have the differential equation

$$\frac{dT}{dt} = k(T - T_1), \quad T(0) = T_0.$$

- Fanciful Example (Example 4, p. 350, textbook): Your kitchen is at $70^\circ$. The temperature of the turkey in your oven is $350^\circ$. One hour after removing the turkey from the oven its temperature is $250^\circ$. What will be its temperature when you serve it 3 hours after removing it from the oven?

$$\frac{d\hat{T}}{dt} = k(\hat{T} - T_1), \quad T > T_1$$

$$\frac{dT}{T-T_1} = k\,dt \quad \int$$

$$\ln(\hat{T}-T_1) = kt + C$$

$$\hat{T}-T_1 = e^C e^{kt}$$

$$\hat{T} = T(t) = T_1 + e^C e^{kt} = T_1 + (T_0-T_1)e^{kt}$$

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\[T(0) = T_i + e^c = T_0\]
\[e^c = T_0 - T_i\]

\[T_0 = 350 \quad T_i = 70\]
\[T_0 - T_i = 280\]

\[T(t) = 70 + 280 \cdot e^{kt}\]

\[T(1) = 250 = 70 + 280 \cdot e^k\]
\[280 \cdot e^k = 180\]
\[e^k = \frac{180}{280}\]
\[k = \ln \left( \frac{180}{280} \right)\]

\[T(3) = 70 + 280 \cdot e^{3 \ln \left( \frac{180}{280} \right)} = 70 + 280 \cdot \left( \frac{180}{280} \right)^3 = 1440\]
• Base $a$ exponential:

$$f(x) = Ca^x$$

where $a > 0$ and $a \neq 1$

$$= C(e^{\ln a})^x$$

$$= Ce^{kx}$$

where $k = \ln a$

• $C = f(0)$ is the **initial value** of $f$.
• $k$ is the **rate constant**.
• If $a > 1$ (and $k > 0$) then we have **exponential growth**.
• If $a < 1$ (and $k < 0$) then we have **exponential decay**.
• Query: what about $k < 0$?
• The defining property of exponential growth and decay is that the growth rate is proportional to the derivative:

$$\frac{d}{dx}f(x) = \frac{d}{dx}Ca^x = \frac{d}{dx}Ce^{kx} = kCe^{kx} = kf(x).$$

• Another way of putting this is that over a time interval of a fixed length $T$ (a year, a day, a century) the function grows by a fixed multiplicative factor:

$$f(t + T) = Ca^{k(t+T)} = a^{kT}Ca^{kt} = a^{kT}f(t).$$
Note that $a^{kT}$ is constant. It does not depend on $t$. It does, of course, depend on the constants $a$ and $T$.

- Yet another way of putting this is that the percentage growth over a fixed time interval is constant.

- The **doubling time** of an exponential function is the (constant) time it takes for the function to double.

- In the case of exponential decay that time would be negative. In the case we speak about the **half life** of the function, the time it takes for the function value to get reduced by a factor $1/2$. 

Examples

Doubling time 1
initial value 200

\[ f(t) = 2^t \cdot 200 \]

Doubling time 3
initial value 50

\[ f(t) = 2^{t/3} \cdot 50 \]

Tripling time 5
i.v. 78

\[ f(t) = 78 \cdot 3^{t/5} = 78 \cdot e^{(\frac{1}{5} \ln 3) t} \]
Compound Interest

Suppose you invest money at \( p \) percent effective annual interest. (Assume some underlying percentage rate is compounded continuously.) What’s the doubling time of your investment?

\[
f(t) = A \cdot \left(1 + \frac{p}{100}\right)^t
\]

\[
\ln \left(1 + \frac{p}{100}\right) = \ln 2
\]

\[
t = \frac{\ln 2}{\ln \left(1 + \frac{p}{100}\right)}
\]

- Doubling time rounded to nearest integer:

\[
\begin{array}{cccccccccccc}
  p & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  DT & : & 70 & 35 & 23 & 18 & 14 & 12 & 10 & 9 & 8 & 7 \\
\end{array}
\]
Bacterial Growth

A population of bacteria starts with a single bacterium with a mass of $10^{-10}$g. (1 pound equals 453g, and one kg equals $10^3$g.) The bacteria split in two, and the population doubles, every 30 minutes (assuming unlimited resources, of course). How long will it take the population to reach a mass of

1 pound ?  
1 pound ?

a person (130lbs) ?

the Earth ($5.972 \times 10^{24}$ kg) ?

the Sun ($2 \times 10^{30}$ kg) ?

the milky way ($2 \times 10^{42}$ kg) ?

the Universe ($10^{53}$kg) ?

• Think about the numbers before going on!

• Letting $t$ denote time measured in hours, and letting $f(t)$ be the mass of the bacteria at time $t$, we get

$$f(t) = 10^{-10} \times 2^{2t} = 10^{-10}4^t.$$  

• We can easily compute the time at which $f(t) = m$:  

\[ t = \frac{\ln (10^{10} m)}{\ln 4} \]

Substituting the appropriate numbers we get:

1 pound: 21 hours
a person (130lbs): 24.5 hours
the Earth \((5.972 \times 10^{24} \text{ kg})\): 62.7 hours
the Sun \((2 \times 10^{30} \text{ kg})\): 72 hours
the milky way \((2 \times 10^{42} \text{ kg})\): 87 hours
the Universe \((10^{53}\text{ kg})\): 110 hours
Logistic Growth

- Of course, exponential growth cannot go on forever (or even for very long)!

- Let \( y(t) \) be the size of a population at time \( t \), and let \( y_0 = y(0) \).

- Suppose the environment has a **carrying capacity** \( L \) that cannot be exceeded.

- The **logistic growth equation** is

\[
y' = ky(L - y), \quad y(0) = y_0 \quad (1)
\]

- The growth rate is always positive, but as the population approaches the carrying capacity the growth rate decreases, and the population never exceeds the carrying capacity.

- We expect the solution to look like:
• Exercise 34, page 354: Show that the solution of (1) is

\[ y(t) = \frac{Ly_0}{y_0 + (L - y_0)e^{-kLt}}. \] (2)

• It’s a great exercise to substitute the solution (2) in (1) and see that the equation is indeed satisfied.

• But let’s solve the differential equation directly.