Notes of 9/2/20

Reminders

- hw 1 closes Friday
- hw 2 is open, closes next week on Friday
- no class on Friday, practice exam. Announcement forthcoming
- no class on Monday, Labor Day

6.4 General Exponentials and Logarithms

- The general (or “base $a$”) exponential function is defined by

$$f(x) = a^x$$

where $a$ is the base and $x$ is the exponent. We assume that

$$a > 0 \quad \text{and} \quad a \neq 1.$$ 

- The inverse of the base $a$ exponential is the base $a$ logarithm defined by

$$a^{\log_a x} = x \quad \text{and} \quad \log_a a^x = x.$$
Do the above 2 equations imply that

\[ a^{\log_a x} = \log_a a^x \, ? \]

- If \( a = e \) we speak of the natural logarithm and write
  
  \[ e^x = \exp x \quad \text{and} \quad \log_e x = \ln x. \]

- WeBWorK (and some other computer languages) assume that \( \log(x) \) denotes the natural logarithm.

- The base 10 logarithm is called the common logarithm.
Figure 1. Some exponentials.

- Figure 1 shows the graphs of the exponential functions with

\[ a = e, 2, 10, 1/e, 1/2, 1/10. \]

Can you tell which is which?
• All exponentials are invertible. The inverse of $f(x) = a^x$ is the base $a$ logarithm.

![Graph of logarithms with base 10, e, 1/10, and 1/e.]

**Figure 2.** Some Logarithms.

• Figure 2 shows the logarithms with base 10, $e$, $1/10$ and $1/e$. Again, can you tell which is which?
Figure 3. Natural Exponential and Logarithm.

- Figure 3 shows the exponential and its inverse, the natural logarithm. As always the graph of the inverse function is the graph of the original function reflected in the line $y = x$ which is also shown.
Properties

- The following key properties of exponentials and logarithms follow straight from the definitions. You should not have to memorize them. Some necessary requirements, like that the input of a logarithm function, or the base of an exponential, be positive, are not mentioned explicitly.

\[
\begin{align*}
    a^x a^y &= a^{x+y} & \log_a (xy) &= \log_a x + \log_a y \\
    \frac{a^x}{a^y} &= a^{x-y} & \log_a \frac{x}{y} &= \log_a x - \log_a y \\
    a^0 &= 1 & \log_a 1 &= 0 \\
    a^{-x} &= \frac{1}{a^x} & \log_a x &= \frac{\ln x}{\ln a} = \frac{\log_b x}{\log_b a} \\
    (ab)^x &= a^x b^x & \log_a x^y &= y \log_a x \\
    \left(\frac{a}{b}\right)^x &= \frac{a^x}{b^x} \\
    (a^x)^y &= a^{xy} \\
    a^x &= \left(e^{\ln a}\right)^x = e^{x \ln a}
\end{align*}
\]
6.5 Exponential Growth and Decay

- Consider again the exponential function

\[ f(t) = a^t = (e^{\ln a})^t = e^{kt} \]

where the rate constant \( k \) is given by

\[ k = \ln a. \]

- Differentiating gives

\[ f'(t) = ke^{kt} = kf(t). \]

- This equation captures the essence of exponential growth. For an exponential function

The growth rate is proportional to the current function value.

- Examples:
  - A population growing at a constant percentage rate
  - Compound Interest
A Fresh Look

• Suppose we start with the differential equation

\[ \frac{dy}{dt} = ky. \]
• Another way of looking at exponential growth: An exponential function grows over a fixed time interval by a fixed factor.
Doubling Time

- The **doubling time** of an exponential function is the amount of time it takes for the function to double. It’s independent of the current function value!

- Example: you invest money at 7% annual interest compounded monthly. How long does it take for your money to double?
Gain Table

The following table shows the factor by which an initial investment is multiplied after $n$ years if it earns an annual interest of $p$ percent.

\[
\text{factor} = \left(1 + \frac{p}{100}\right)^n
\]
<table>
<thead>
<tr>
<th>$p$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.1</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.3</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>1.0</td>
<td>1.1</td>
<td>1.1</td>
<td>1.2</td>
<td>1.2</td>
<td>1.3</td>
<td>1.3</td>
<td>1.4</td>
<td>1.4</td>
<td>1.5</td>
<td>1.5</td>
<td>1.6</td>
</tr>
<tr>
<td>6</td>
<td>1.0</td>
<td>1.1</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
<td>1.3</td>
<td>1.4</td>
<td>1.5</td>
<td>1.6</td>
<td>1.7</td>
<td>1.8</td>
<td>1.9</td>
<td>2.0</td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
<td>1.4</td>
<td>1.5</td>
<td>1.6</td>
<td>1.7</td>
<td>1.9</td>
<td>2.0</td>
<td>2.1</td>
<td>2.3</td>
<td>2.5</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>1.1</td>
<td>1.2</td>
<td>1.3</td>
<td>1.5</td>
<td>1.6</td>
<td>1.8</td>
<td>2.0</td>
<td>2.2</td>
<td>2.4</td>
<td>2.6</td>
<td>2.8</td>
<td>3.1</td>
</tr>
<tr>
<td>12</td>
<td>1.0</td>
<td>1.1</td>
<td>1.3</td>
<td>1.4</td>
<td>1.6</td>
<td>1.8</td>
<td>2.0</td>
<td>2.3</td>
<td>2.5</td>
<td>2.8</td>
<td>3.1</td>
<td>3.5</td>
<td>3.9</td>
</tr>
<tr>
<td>14</td>
<td>1.0</td>
<td>1.1</td>
<td>1.3</td>
<td>1.5</td>
<td>1.7</td>
<td>2.0</td>
<td>2.3</td>
<td>2.6</td>
<td>2.9</td>
<td>3.3</td>
<td>3.8</td>
<td>4.3</td>
<td>4.9</td>
</tr>
<tr>
<td>16</td>
<td>1.0</td>
<td>1.2</td>
<td>1.4</td>
<td>1.6</td>
<td>1.9</td>
<td>2.2</td>
<td>2.5</td>
<td>3.0</td>
<td>3.4</td>
<td>4.0</td>
<td>4.6</td>
<td>5.3</td>
<td>6.1</td>
</tr>
<tr>
<td>18</td>
<td>1.0</td>
<td>1.2</td>
<td>1.4</td>
<td>1.7</td>
<td>2.0</td>
<td>2.4</td>
<td>2.9</td>
<td>3.4</td>
<td>4.0</td>
<td>4.7</td>
<td>5.6</td>
<td>6.5</td>
<td>7.7</td>
</tr>
<tr>
<td>20</td>
<td>1.0</td>
<td>1.2</td>
<td>1.5</td>
<td>1.8</td>
<td>2.2</td>
<td>2.7</td>
<td>3.2</td>
<td>3.9</td>
<td>4.7</td>
<td>5.6</td>
<td>6.7</td>
<td>8.1</td>
<td>9.6</td>
</tr>
<tr>
<td>22</td>
<td>1.0</td>
<td>1.2</td>
<td>1.5</td>
<td>1.9</td>
<td>2.4</td>
<td>2.9</td>
<td>3.6</td>
<td>4.4</td>
<td>5.4</td>
<td>6.7</td>
<td>8.1</td>
<td>9.9</td>
<td>12.1</td>
</tr>
<tr>
<td>24</td>
<td>1.0</td>
<td>1.3</td>
<td>1.6</td>
<td>2.0</td>
<td>2.6</td>
<td>3.2</td>
<td>4.0</td>
<td>5.1</td>
<td>6.3</td>
<td>7.9</td>
<td>9.8</td>
<td>12.2</td>
<td>15.2</td>
</tr>
<tr>
<td>26</td>
<td>1.0</td>
<td>1.3</td>
<td>1.7</td>
<td>2.2</td>
<td>2.8</td>
<td>3.6</td>
<td>4.5</td>
<td>5.8</td>
<td>7.4</td>
<td>9.4</td>
<td>11.9</td>
<td>15.1</td>
<td>19.0</td>
</tr>
<tr>
<td>28</td>
<td>1.0</td>
<td>1.3</td>
<td>1.7</td>
<td>2.3</td>
<td>3.0</td>
<td>3.9</td>
<td>5.1</td>
<td>6.6</td>
<td>8.6</td>
<td>11.2</td>
<td>14.4</td>
<td>18.6</td>
<td>23.9</td>
</tr>
<tr>
<td>30</td>
<td>1.0</td>
<td>1.3</td>
<td>1.8</td>
<td>2.4</td>
<td>3.2</td>
<td>4.3</td>
<td>5.7</td>
<td>7.6</td>
<td>10.1</td>
<td>13.3</td>
<td>17.4</td>
<td>22.9</td>
<td>30.0</td>
</tr>
<tr>
<td>32</td>
<td>1.0</td>
<td>1.4</td>
<td>1.9</td>
<td>2.6</td>
<td>3.5</td>
<td>4.8</td>
<td>6.5</td>
<td>8.7</td>
<td>11.7</td>
<td>15.8</td>
<td>21.1</td>
<td>28.2</td>
<td>37.6</td>
</tr>
<tr>
<td>34</td>
<td>1.0</td>
<td>1.4</td>
<td>2.0</td>
<td>2.7</td>
<td>3.8</td>
<td>5.3</td>
<td>7.3</td>
<td>10.0</td>
<td>13.7</td>
<td>18.7</td>
<td>25.5</td>
<td>34.8</td>
<td>47.1</td>
</tr>
<tr>
<td>36</td>
<td>1.0</td>
<td>1.4</td>
<td>2.0</td>
<td>2.9</td>
<td>4.1</td>
<td>5.8</td>
<td>8.1</td>
<td>11.4</td>
<td>16.0</td>
<td>22.3</td>
<td>30.9</td>
<td>42.8</td>
<td>59.1</td>
</tr>
<tr>
<td>38</td>
<td>1.0</td>
<td>1.5</td>
<td>2.1</td>
<td>3.1</td>
<td>4.4</td>
<td>6.4</td>
<td>9.2</td>
<td>13.1</td>
<td>18.6</td>
<td>26.4</td>
<td>37.4</td>
<td>52.8</td>
<td>74.2</td>
</tr>
<tr>
<td>40</td>
<td>1.0</td>
<td>1.5</td>
<td>2.2</td>
<td>3.3</td>
<td>4.8</td>
<td>7.0</td>
<td>10.3</td>
<td>15.0</td>
<td>21.7</td>
<td>31.4</td>
<td>45.3</td>
<td>65.0</td>
<td>93.1</td>
</tr>
<tr>
<td>42</td>
<td>1.0</td>
<td>1.5</td>
<td>2.3</td>
<td>3.5</td>
<td>5.2</td>
<td>7.8</td>
<td>11.6</td>
<td>17.1</td>
<td>25.3</td>
<td>37.3</td>
<td>54.8</td>
<td>80.1</td>
<td>116.7</td>
</tr>
<tr>
<td>44</td>
<td>1.0</td>
<td>1.5</td>
<td>2.4</td>
<td>3.7</td>
<td>5.6</td>
<td>8.6</td>
<td>13.0</td>
<td>19.6</td>
<td>29.6</td>
<td>44.3</td>
<td>66.3</td>
<td>98.7</td>
<td>146.4</td>
</tr>
<tr>
<td>46</td>
<td>1.0</td>
<td>1.6</td>
<td>2.5</td>
<td>3.9</td>
<td>6.1</td>
<td>9.4</td>
<td>14.6</td>
<td>22.5</td>
<td>34.5</td>
<td>52.7</td>
<td>80.2</td>
<td>121.6</td>
<td>183.7</td>
</tr>
<tr>
<td>48</td>
<td>1.0</td>
<td>1.6</td>
<td>2.6</td>
<td>4.1</td>
<td>6.6</td>
<td>10.4</td>
<td>16.4</td>
<td>25.7</td>
<td>40.2</td>
<td>62.6</td>
<td>97.0</td>
<td>149.8</td>
<td>230.4</td>
</tr>
<tr>
<td>50</td>
<td>1.0</td>
<td>1.6</td>
<td>2.7</td>
<td>4.4</td>
<td>7.1</td>
<td>11.5</td>
<td>18.4</td>
<td>29.5</td>
<td>46.9</td>
<td>74.4</td>
<td>117.4</td>
<td>184.6</td>
<td>289.0</td>
</tr>
</tbody>
</table>

You want to invest early and wisely!
Example 2, textbook: Suppose you are growing a culture of bacteria. You have 10,000 bacteria at noon and 40,000 at 2pm. How many will you have at 5pm?


Radioactive Dating

- not everything grows.

- **exponential decay**: measured by **half life**, the time required to reduce a given amount or population by one half.

- Example: the half life of Carbon 14 ($^{14}C$) is 5,730 year.

- Try again
  
  \[ f(t) = e^{kt} \]

- We want \( k \) to be such that
  
  \[ e^{5730k} = \frac{1}{2} \]

- This equation can be solved easily

  \[ k = \frac{\ln 1/2}{5730} \approx -0.000121. \]

- Problem 17 on p. 353 of the textbook describes the basis of radioactive dating. The procedure was worked out by Willard Libby in the 1940s. Libby later received the Nobel Prize for his work.

- Carbon 12, ($^{14}C$), is stable. It does not decay (on the time scales of interest). Carbon 14
does decay, but it is also replenished by the interaction of cosmic rays with the Carbon 14 in the atmosphere. As a result, the ratio of

\[ R = \frac{14C}{12C} \]

in the atmosphere is essentially constant.

- Living Organisms interact with the atmosphere and that ratio in the organism’s body is the same as in the atmosphere.
- When the organism dies the interaction with the atmosphere stops, \(14C\) in the organism continues to decay, and the ratio \(R\) decreases. By measuring the ratio we can estimate (quite accurately) the length of time that has passed since the organism died.
• Example: The ratio $R$ in the hair of a frozen mammoth found in the Siberian Tundra is 30% of its ordinary value. How long ago did the mammoth die?

• Many variations of this technique are available and allow scientists to determine the ages of many minerals and other material.
Newton’s Law of Cooling

• The rate at which an object cools or warms is proportional to the difference of the temperature of the object and the temperature of the surrounding medium. Let $T(t)$ denote the temperature of the object at time $t$ and $T_1$ the constant temperature of the surrounding medium.

• Then we have the differential equation

$$\frac{dT}{dt} = k(T - T_1), \quad T(0) = T_0.$$ 

• Fanciful Example (Example 4, p. 350, textbook): Your kitchen is at $70^\circ$. The temperature of the turkey in your oven is $350^\circ$. One hour after removing the turkey from the oven its temperature is $250^\circ$. What will be its temperature when you serve it 3 hours after removing it from the oven?