Reminders

- DS today at 5:00pm
- No DS Thursday or Friday
- hw 14 all review
- next week all review
- Final Exam December 11-12 (Friday-Saturday).
• Suppose we have the polar equation

\[ r = f(\theta). \]

• For example, \( f(\theta) = 1 \).

• or \( f(\theta) = \theta \)

• Question in yesterday’s chat room: how can these be functions. The graphs fail the vertical line test.
- Recall our area formula

\[ A = \frac{1}{2} \int_{\alpha}^{\beta} f^2(\theta) \, d\theta. \]
Recall our computation of the area $A$ of the circle

$$f(\theta) = \sin \theta.$$ 

\[ A = \frac{1}{2} \int_0^\pi \sin^2 \theta \, d\theta = \frac{\pi}{4}. \]
Figure 2. Flower, $r = 4 \sin(2\theta)$.

- Example 2 textbook. Compute the area of one leaf of the curve defined by

$$r = 4 \sin(2\theta) \quad \text{where} \quad 0 \leq \theta \leq \frac{\pi}{2}.$$ 

- Figure 2 shows the whole curve, with

$$0 \leq \theta \leq 2\pi.$$
$$A = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} f^2(\theta) \, d\theta = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} 16 \cdot \sin^2(2\theta) \, d\theta$$

ex.: $$\int_{0}^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta = \frac{\pi}{4}$$

$$A = \frac{1}{2} \cdot 16 \cdot \frac{\pi}{4} = 2\pi \approx 6.28$$
Figure 3. $r = 4 \sin(6\theta)$.

- For the fun of it: what is the area of one petal of this flower?

\[
A = \frac{1}{2} \int_0^{\pi/6} 16 \sin^2(6\theta) \, d\theta
\]

\[
= \frac{1}{2} \cdot 16 \cdot \frac{\pi}{12} = \frac{2\pi}{3}
\]
Derivatives

• Again, suppose
  \[ r = f(\theta). \]

• We can of course easily compute
  \[ f'(\theta) = \frac{d}{d\theta} f(\theta) \]

• This would be the same as we did before except that instead of \( y \) and \( x \) we use \( r \) and \( \theta \). \( f'(\theta) \) tells us how quickly \( f \) changes as \( \theta \) changes.

• But suppose we want to find the slope of a tangent, i.e., we want \( \frac{dy}{dx} \).

• Recall that
  \[ x = r \cos \theta = f(\theta) \cos \theta \]
  and
  \[ y = r \sin \theta = f(\theta) \sin \theta. \]

• We get
  \[
  \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.
  \]

• Let’s check the formula with a circle:
\[
\begin{align*}
  x &= \cos \theta = r \cos \theta \\
  y &= \sin \theta = r \sin \theta \\
  x^2 + y^2 &= 1 \\
  2x + 2yy' &= 0 \\
  y' &= -\frac{x}{y} \\
  \frac{dy}{dx} &= \frac{0 \cdot \sin \theta + 1 \cdot \cos \theta}{0 \cdot \cos \theta - 1 \cdot \sin \theta} = \frac{\cos \theta}{\sin \theta} = -\frac{x}{y}
\end{align*}
\]
• Example: Return to the **Spiral of Archimedes**

\[ r = f(\theta) = \theta. \]

\[ f'(\theta) = 1 \]

In what direction does it start?

• Recall

\[
\frac{dy}{d\theta} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}.
\]

\[
= \frac{\sin \theta + \theta \cos \theta}{\cos \theta - \theta \sin \theta} = 0
\]

\[
\left. \frac{dy}{dx} \right|_{\theta=0} = 0
\]
Figure 4. Beginning of Spiral of Archimedes.
Kepler’s Laws

- Problem 32, page 551, textbook
- Johannes Kepler 1571-1630

- Kepler found his three laws empirically, by using (his teacher Brahe’s) observations, guessing the law, and confirming consistency with the observations.

- His work would have been a lot easier had he known Calculus!
• Kepler’s Laws are:

1. The orbit of a planet is an ellipse, with the Sun at one of its foci.

2. A line segment joining a planet and the Sun sweeps out equal areas in equal time intervals. (Thus the planet moves faster when it is closer to the Sun.)

3. The square of the orbital period of the plane is proportional to the cube of the semi-major axis of its orbit.
• Let’s focus on the second law. Suppose
\[ \theta = \theta(t) \]
is the angle made by the line from the planet to the Sun at time \( t \) and
\[ r = f(\theta) \]
is the distance of the planet from the Sun.

• We will use the physical fact that the angular momentum
\[ M = mr^2 \frac{d\theta}{dt} \]
is constant.

• Let’s consider the angles
\[ \alpha = \theta(t_1) \quad \text{and} \quad \beta = \theta(t_2) \]
and consider the area \( A \) swept out in the time interval \([t_1, t_2]\). It is given by
\[
A = \frac{1}{2} \int_{\alpha}^{\beta} r^2(\theta) d\theta \\
= \frac{1}{2} \int_{t_1}^{t_2} r^2(\theta) \frac{d\theta}{dt} dt \\
= \frac{1}{2} \int_{t_1}^{t_2} M \frac{dt}{m} \\
= \frac{1}{2} \frac{M}{m} (t_2 - t_1)
\]
i.e., the area swept out during the time interval \([t_1, t_2]\) is indeed constant.
• We are done with Chapter 10.

• As previously announced, we’ll spend the remaining time after Thanksgiving on reviewing the entire semester.

Happy Thanksgiving!