Review continued ...

- We usually compute definite integrals via the Fundamental Theorem of Calculus:

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \quad \text{where} \quad F' = f. \]

- However, some integrals can be computed without knowing an antiderivative. In Math 1210, we discussed in particular:

\[ \int_{-c}^{c} f(x) \, dx = 0 \quad \text{if} \quad f \quad \text{is odd,} \quad (1) \]

\[ \int_{a}^{a+2\pi} \sin^2 x \, dx = \int_{a}^{a+2\pi} \cos^2 x \, dx = \pi, \quad (2) \]

and

\[ \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx = \frac{\pi r^2}{2}. \quad (3) \]

- On the final exam for Math 1210 it was a common misconception that the integral (3) was zero, because of the symmetry. This can’t possibly be true since the integrand is never negative.
• The **Mean Value Theorem for Integrals**: There is a point $c$ in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b - a). \quad (4)$$

• You can be casual when computing an antiderivative, once you have it check it by differentiation.

• We used definite integrals to solve the following problems:
  - Computation of areas
  - Computation of volumes by integrating the area of the cross section, using the methods of slabs, disks, washers, or shells.
  - Computation of the length of a plane curve.
  - Computation of the surface area of a solid of revolution
  - Computation of Work.
  - Computation of the center of mass.

• The limits of integration may depend on a variable. We can differentiate with respect to that variable without actually computing an antiderivative:

$$\frac{d}{dx} \int_{L(x)}^{U(x)} f(t)dt = \frac{d}{dx} \left( F(U(x)) - F(L(x)) \right)$$

$$= f(U(x))U'(x) - f(L(x))L'(x) \quad (5)$$
where, as usual, $F$ is any antiderivative of $f$.

- A special case of that formula is this version of the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_{a}^{x} f(t)dt = f(x). \quad (6)$$

- The Fundamental Theorem of Calculus says that differentiation and integration are inverse processes of each other.

- Following is a list of some words and phrases, listed in alphabetical order, that you should be able to define, use, and understand.

  acceleration, antiderivative, asymptote, base, chain rule, concave down, concave up, constant, continuity, critical points, cubic, decreasing function, definite integral, degree of a polynomial, denominator, dependent variable, derivative, differential, differential equation, domain, even function, equation, exponent, expression, first derivative, function, Fundamental Limit Theorem, Fundamental Theorem of Calculus, graph (of a function or an equation), implicit differentiation, increasing function, indefinite integral, independent variable, inflection point, integrand, integration constant, integration variable, Leibniz notation, limit, limits of integration (upper and lower), linear, Mean Value Theorem for derivatives, Mean Value Theorem for integrals, method of disks, method of shells, method of slabs, method
of washers, Newton’s method, numerator, odd functions, points of inflection, polynomial, position, power rule, power, product rule, quadratic, quartic, quintic, quotient rule, radical, range, rational function, related rates, Riemann Sum, secant, second derivative, singular point, solid of revolution, stationary point, sum rules, tangent, velocity, work.

- **Contents of Math 1220** More differentiation and integration rules (particularly exponentials, logarithms, inverse trig functions, integration by parts, logarithmic differentiation), more applications, indeterminate expressions, improper integrals, sequences and series (particularly their convergence, power series, Taylor series).
6.1 The natural logarithmic function

- We are now starting with the subject of Math 1220 proper.
- I assume you are familiar with exponentials and logarithms, particularly the natural logarithm.

\[ \ln, \quad e^x, \quad e = 2.71828\ldots \]

- The major fact that we will begin to understand today is

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

or, equivalently,

\[
\int_1^x \frac{1}{t} \, dt = \ln x.
\]

- This makes geometric sense, as illustrated in Figure 1.
- Note that our power rule

\[
\int t^r \, dt = \frac{t^{r+1}}{r+1}
\]

fails if \( r = -1 \). We will see today that

\[
\int_1^x t^{-1} \, dt = \int_1^x \frac{1}{t} \, dt = \ln x.
\]
Figure 1. \( \ln \) and its derivative.

- There are many different ways to introduce the derivatives of exponentials and logarithms. In the spirit of Math 1210 we could define

\[
\frac{d}{dx} \ln x = \lim_{h \to 0} \frac{\ln(x + h) - \ln x}{h}.
\]

- It would be hard to compute that limit with what we have learned in Math 1210.

- Instead the textbook proceeds as follows: It
defines

\[ \ln x = \int_1^x \frac{1}{t} \, dt. \]

- With that definition it follows immediately from the FToC that

\[ \frac{d}{dx} \ln x = \frac{1}{x}. \]

- Of course, this definition is useful only if it is consistent with our already existing understanding of the natural logarithm!

- It is, and we will eventually recognize that this is so by showing that the function we defined has all the properties if the natural logarithm with which we are familiar.
• Verifying a bunch of properties of the logarithm will give us the opportunity to practice the differentiation of logarithmic function.

• First example:

\[
\frac{d}{dx} \ln |x| = \frac{1}{x}.
\]

\[
\text{clear if } x > 0
\]

\[
x < 0 \quad \frac{d}{dx} \ln(-x) = \frac{-1}{-x} = \frac{1}{x} < 0
\]
• Next verify that the following properties of the natural logarithm are satisfied.

\[
\begin{align*}
\ln 1 &= 0 \quad \text{(i)} \\
\ln ab &= \ln a + \ln b \quad \text{(ii)} \\
\ln \frac{a}{b} &= \ln a - \ln b \quad \text{(iii)} \\
\ln a^r &= r \ln a \quad \text{(iv)}
\end{align*}
\]

(assuming that \(a, b > 0\)).

\[(i) \quad \ln 1 = 0 = \int_1^1 \frac{1}{t} \, dt\]
(ii) \( \ln ab = \ln a + \ln b. \)

\[
\begin{align*}
  f(x) &= \ln(ax) \\
  f'(x) &= \frac{a}{ax} = \frac{1}{x} = \frac{d}{dx} \ln x \\
  \ln ax &= \ln x + C' \\
  \ln a &= \ln 1 + C' \\
  &= C' \\
  \ln (ax) &= \ln a + \ln x
\end{align*}
\]
\[ (iii) \quad \ln \frac{a}{b} = \ln a - \ln b. \]

\[ \frac{a}{b} = a \cdot \frac{1}{b} \]

\[ \ln \frac{1}{b} = -\ln b \]

\[ f(x) = \ln \frac{1}{x} \]

\[ f'(x) = -\frac{1}{x} \cdot \frac{1}{x^2} = -\frac{1}{x^2} = -\frac{d}{dx} \ln x \]

\[ \ln \frac{1}{x} = -\ln x + C \]

\[ x = 1 \]
\((iv) \quad \ln a^r = r \ln a.\)

\(r \quad \text{nat. #}\)

\[\ln a^r = \ln a \cdot a \cdot \ldots \cdot a = \ln a + \ln a + \ldots + \ln a = r \ln a\]

\[f(x) = \ln x^r\]

\[f'(x) = \frac{r x^{r-1}}{x^r} = \frac{r}{x} = r \frac{d}{dx} \ln x\]

\[f(x) = r \ln x + c \quad \Rightarrow \quad c' = 0\]
Examples

Compute \( \frac{d}{dx} \ln x^2 = ? \)
in two different ways.

\[
\frac{d}{dx} \ln x^2 = \frac{2x}{x^2} = \frac{2}{x}
\]

\[
\frac{d}{dx} \ln x^2 = \frac{d}{dx} \ln (2x) = 2 \frac{d}{dx} \ln x = \frac{2}{x}
\]
\[ \int \frac{2x}{1+ x^2} \, dx = ? \quad \ln(1+x^2) \quad CBD \]

Ex.: \( u = 1 + x^2 \)
\[
\frac{d}{dx} x^x =? \\

\begin{align*}
\ln y &= x \ln x \\
\frac{y'}{y} &= \ln x + x \cdot \frac{1}{x} = \ln x + 1 \\
y' &= y(\ln x + 1) \\
&= x^x(\ln x + 1)
\end{align*}

\text{logarithmic differentiation}
\[ \int \tan x \, dx = ? \]

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \ln |\cos x| \]
\[ \int \frac{x^2 + 1}{x + 1} \, dx = ? \]