Math 1220-3, 7, 90 — Fall 2020

Notes of 11/16/20

Announcements

• DS today at 3:00 (Scott).

• Exam 3 on chapter 9 will take place on Friday and Saturday, November 20–21. It will have 8 questions on Chapter 9, including Taylor and MacLaurin Series.

• We will spend the beginning of this week on reviewing chapter 9. Today I will go over these notes, and you are welcome to ask questions as we go along. Tuesday and Wednesday we will have more review, but it will be driven by your questions.

• We will have class on Friday. We will not cover new material, but Liz and I will answer questions about anything other than Chapter 9.

Review of Chpt 9

• As usual, the list is not complete and not self contained. Each point should remind you of the relevant facts and concepts, and trigger your understanding. Don’t try to memorize this list. Instead, understand the subject and the many connections among the facts and concepts.
• A sequence is of the form

\[ a_0, a_1, a_2, a_3, \ldots \] (1)

where the terms of the sequence \( a_n \) are real numbers. An industrial strength mathematical definition of a sequence is that it is a function whose domain is the set of natural numbers, and whose range is the set of real numbers.

• The textbook uses the phrase infinite sequence, we usually drop the word infinite.

• A sequence can be specified in various ways, e.g., by giving enough initial terms to establish a pattern, a recursion formula, or an explicit formula for the \( n \)-th term.

• We say

\[ L = \lim_{n \to \infty} a_n, \] (2)

or the sequence converges to \( L \), if for all \( \varepsilon > 0 \) there exists a number \( N \) such that

\[ n > N \implies |a_n - L| < \varepsilon. \] (3)

• If the sequence has a (finite) limit we say it converges, otherwise it diverges. We might say a sequence converges to infinity, but in that case it actually diverges.

• Squeeze Theorem. Suppose that the sequences \( \{a_n\} \) and \( \{c_n\} \) both converge to \( L \), and that \( a_n \leq b_n \leq c_n \) for \( n \geq K \), \( K \) being
some fixed integer. Then \( \{b_n\} \) also converges to \( L \).

- **bounded sequences.** A number \( U \) is an **upper bound** for a sequence \( \{a_n\} \) if \( a_n \leq U \) for all \( n \). Similarly, a number \( L \) is a **lower bound** for a sequence \( \{a_n\} \) if \( a_n \geq L \) for all \( n \). Note that if \( U \) is an upper bound, then any larger number is also an upper bound. Similarly for lower bounds.

- A sequence \( \{a_n\} \) is **monotonic** if it is non-increasing, or non-decreasing. It is **non-increasing** if \( a_{n+1} \leq a_n \) for all \( n \), and it is **non-decreasing** if \( a_{n+1} \geq a_n \) for all \( n \).

- **Monotonic Sequence Theorem.** This is a deep result! A non-increasing sequence that is bounded below by a number \( L \) converges to a limit that is greater than or equal to \( L \). Similarly, a non-decreasing sequence that is bounded above by a number \( U \) converges to a limit that is less than or equal to \( U \).

- **Finitely many exceptions don’t matter.** The behavior of the first finitely many terms of any sequence do not affect its convergence or divergence, or its limit.

- The **\( n \)-th partial sum** of a sequence \( \{a_n\} \), \( n = 0, 1, 2, \ldots \) is defined by

\[
S_n = \sum_{k=0}^{n} a_k. \tag{4}
\]
• If \( \{a_n\} \) is an infinite sequence its \textbf{infinite series} is
\[
\sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} S_n.
\]
(5)

As for sequences, we usually omit the word \textit{infinite}. (A \textbf{finite series} is just a plain sum, we don’t need a new word for that.) The \textbf{series converges} if the limit of its partial sums exists, it \textbf{diverges} otherwise.

• \textbf{\( n \)-th term divergence test.} If a series \( \sum_{n=0}^{\infty} a_n \) converges then \( \lim_{n \to \infty} a_n = 0 \). Contrapositively, if the sequence \( \{a_n\} \) does not converge to zero then the series diverges.

• However, a series may diverge even if its terms converge to zero. A very major example is the \textbf{harmonic series}:
\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \rightarrow \infty.
\]
(6)

\textbf{The harmonic series diverges} (barely). We can see this, for example, by the integral test, or by finding infinitely many subsets of terms that add to greater than 1/2:

\[
\underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \ldots + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \ldots + \frac{1}{16}}_{\geq \frac{1}{2}}
\]
\[
+ \underbrace{\frac{1}{17} + \ldots + \frac{1}{32}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{33} + \ldots + \frac{1}{64}}_{\geq \frac{1}{2}} + \ldots
\]
(7)
• In plain English: **Do not confuse the convergence of a series with the convergence of its terms.** If the series converges then the sequence of its terms converges to zero. If the terms of a series converge to zero the series itself may or may not converge, depending on its precise definition.

• A sequence is **geometric** if the quotient of successive terms is constant. The associated series is a **geometric series**. Its behavior is governed by the (easily verified) equation

\[
\sum_{k=1}^{n+1} r^{k-1} = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}. \tag{8}
\]

Thus \(\sum r^k\) converges if and only if \(|r| < 1\). In that case

\[
\sum_{k=1}^{\infty} r^{k-1} = \sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}. \tag{9}
\]

• **Bounded Sum Test.** A series of non-negative terms converges if and only if its partial sums are bounded above.

• **Integral Test.** Let \(f\) be a continuous, positive, non-increasing function on \([1, \infty)\) such that \(a_k = f(k)\) for \(k = 1, 2, 3, \ldots\). Then the series \(\sum_{k=1}^{\infty} a_n\) converges if and only if the improper integral \(\int_{1}^{\infty} f(x)dx\) converges.

• Of course, finitely many exceptions don’t matter, so in the above integral the lower limit
could be replaced with any arbitrarily large positive number, and \( a_k = f(k) \) needs to be valid only from that number onward.

- **\( p \)-series test** (or just \( p \)-test). The series

\[
\sum_{k=1}^{\infty} \frac{1}{k^p}
\]  

(10)

converges if \( p > 1 \) and diverges if \( p \leq 1 \).

- **Ordinary Comparison Test.** Suppose that \( 0 \leq a_n \leq b_n \) for \( n \geq N \). If \( \sum b_n \) converges, then so does \( \sum a_n \). If \( \sum a_n \) diverges, then so does \( \sum b_n \).

- **Limit Comparison Test.** Suppose that \( a_n \geq 0 \), \( b_n > 0 \), and

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = L.
\]  

(11)

If \( 0 < L < \infty \) then either both series converge, or both series diverge. If \( L = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.

- **Ratio Test.** Let \( \sum a_n \) be a series of positive terms and suppose that

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho.
\]  

(12)

If \( \rho < 1 \), the series converges. If \( \rho > 1 \), the series diverges. If \( \rho = 1 \) the test is inconclusive.
• **Alternating Series Test.** Let

\[ a_1 - a_2 + a_3 - a_4 + \ldots \quad (13) \]

be an alternating series with \( a_n > a_{n+1} > 0 \). If \( \lim_{n \to \infty} a_n = 0 \), then the series converges.

• We say the series \( \sum a_n \) **converges absolutely** if \( \sum |a_n| \) converges.

• If a series converges absolutely it converges.

• A convergent series may not converge absolutely. Example: the alternating harmonic series.

• **Absolute Ratio Test.** Let \( \sum u_n \) be a series of non-zero terms and suppose that

\[
\lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \rho. \quad (14)
\]

If \( \rho < 1 \), the series converges absolutely. If \( \rho > 1 \), the series diverges. If \( \rho = 1 \) the test is inconclusive.

≥ Fascinating tidbit: **Rearrangement Theorem.** The terms of an absolutely convergent series can be rearranged arbitrarily without affecting convergence or the limit. On the other hand, the terms of the conditionally convergent harmonic series can be rearranged to get any limit whatsoever, or a divergent series. In class we discussed how to rearrange the alternating harmonic series to obtain any limit we want. Let’s
call it $L$ and assume without loss of generality that $L > 0$. Then start by adding the first few positive terms of the alternating harmonic series until you get a partial sum greater than $L$ for the first term. Add the first negative term. Your new partial sum is less than $L$. Add enough positive term just to get above $L$. Add the next negative term. Continue in this fashion.

- Power series are generalizations of polynomials (from finitely many to infinitely many terms). A power series is of the form

$$S(x) = \sum_{n=0}^{\infty} a_n x^n. \quad (15)$$

- The convergence set of a power series is the set of $x$ for which the series converges. The convergence set is always an interval of one of these three types:
  - The single point $x = 0$.
  - An interval $(-R, R)$, plus possibly one or both endpoints. $R$ is the radius of convergence, the interval is the interval of convergence.
  - The set of all real numbers.

- A power series converges absolutely on the interior of its interval of convergence.

- Shift of Origin. The origin can be shifted to give a series of the form $\sum a_n (x - a)^n$. 

• Convergent power series can be processed much like polynomials, e.g., they can be combined by arithmetic operations, differentiated, and integrated.

• In what follows let $f$ be an arbitrarily often differentiable function.

• The Taylor Series of $f$ about $a$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.$$  \hfill (16)

• If $a = 0$ the Taylor Series is also called a MacLaurin Series.

• The Taylor series is defined by the requirement that

$$T^{(n)}(a) = f^{(n)}(a), \quad n = 0, 1, 2, \ldots$$  \hfill (17)
Some Taylor Series

- Some important MacLaurin Series that we have calculated are:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \]

\[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \]

\[ \frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \]

\[ \ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \frac{(x - 1)^5}{5} - \ldots \]
• The $n$-th partial sum of the Taylor series is the $n$-th degree (or, sometimes, order) **Taylor Polynomial**.

• Taylor’s **Formula** says:

\[
f(x) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \quad (18)
\]

where

\[
R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \quad (for \ some \ c \ in \ (a,x)) \quad (19)
\]

• Clearly, a Taylor series converges if and only if its remainder converges to zero.

• An alternative form of the remainder is

\[
R_n(x) = \int_{a}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt. \quad (20)
\]

This can be seen easily by repeated integra-
tion by parts:

\[ f(x) = f(a) + \int_a^x 1 \times f'(t)dt \]

\[ = f(a) + (t - x)f'(t)|_a^x - \int_a^x (t - x)f''(t)dt \]

\[ = f(a) + (x - a)f'(a) - \frac{1}{2}(t - x)^2 f''(t)|_a^x + \]

\[ + \frac{1}{2} \int_a^x (t - x)^2 f'''(t)dt \]

\[ = f(a) + (x - a)f'(a) + \frac{1}{2}(x - a)^2 f''(a) + \]

\[ + \frac{1}{2} \left( \frac{1}{3}(t - x)^3 f'''(t)|_a^x - \frac{1}{3} \int_a^x (t - x)^3 f''''(t)dt \right) \]

\[ = \ldots \]

(21)
• Euler’s Formula

\[ e^{i\theta} = \cos \theta + i \sin \theta \quad \text{where} \quad i^2 = -1. \]  

(22)

is obtained by expanding the complex exponential into its MacLaurin Series and separating real and imaginary terms. Evaluating at \( \theta = \pi \) and rearranging gives the famous and beautiful equation

\[ e^{i\pi} + 1 = 0 \]  

(23)

combining the four most important numbers in mathematics, \( e, \pi, 0, \) and 1, in one simple equation.

• One application of Taylor series is the approximation of functions. For example, to approximate \( e^x \) in the interval \([-1, 1]\) by its MacLaurin polynomial with an accuracy \( \epsilon \), say, we pick \( n \) such that

\[
\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| = \left| e^c \frac{x^{n+1}}{(n+1)!} \right| < \left| \frac{3x^{n+1}}{(n+1)!} \right| < \frac{3}{(n+1)!} < \epsilon. 
\]  

(24)

• Another application of Taylor Series is the solution of differential equations. For example, if

\[ y' = y^2, \quad y(0) = 1, \]  

(25)

we can compute that

\[ y'' = 2yy', \quad y''' = 2(y'^2 + yy''), \quad \ldots \]  

(26)
and

\[ y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 2, \quad y'''(0) = 6, \quad \ldots, \quad y^n(0) = n!, \quad (27) \]

and hence

\[ y(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \quad |x| < 1. \quad (28) \]

• Yet another application we discussed in class is the solution of nonlinear equations. If \( f(x, y) \) is an expression in \( x \) and \( y \), we think of \( y \) as a function of \( x \), we know that \( y(x_0) = x_0 \), and we want to solve

\[ f(x, y) = 0 \quad (29) \]

then we can differentiate repeatedly and implicitly in (29), evaluate at \((x_0, y_0)\), and compute the first few derivatives of \( y \) at \( x_0 \). We can then use these derivatives to construct a Taylor polynomial. If we can recognize a pattern and evaluate all derivatives at \((x_0)\) we obtain the complete Taylor series and hence the whole function. (This is a simplification, I am ignoring the issue of convergence and the sum of the Taylor series.)