History is in the making and regardless of political orientation the ongoing election is highly emotional for many people. The University offers resources to deal with the current situation. See
https://attheu.utah.edu/facultystaff/checkonyourucrew-post-election-support/
for more information.

Reminders

• DS today 5:00 and tomorrow 1:00 and 5:00.

• For the remainder of the semester, hws close on Sundays.
Review of some Notation

- Recall that $f^{(n)}$ denotes the $n$-th derivative of $f$.

- $n!$ is called “$n$-factorial” and is defined by

$$n! = \begin{cases} 
1 & \text{if } n = 0 \\
1 \times 2 \times \ldots \times n & \text{if } n > 0 
\end{cases}$$

- The first few factorials are given in this Table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n!$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
<td>5,040</td>
<td>40,320</td>
<td>362,880</td>
</tr>
</tbody>
</table>

- The factorial grows faster than any exponential!

- A few useful factorial identities are:

$$n! = (n-1)!n,$$

$$\frac{n!}{(n-1)!} = n,$$

$$\frac{n!}{n} = (n-1)!,$$

$$\frac{n!}{(n-2)!} = n(n-1),$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- The last expression, $\binom{n}{k}$, is pronounced “$n$ choose $k$” and denotes the number of ways you can choose $k$ objects out of $n$ objects.
• For example, each NBA team can have a maximum of 15 players, 13 of which can be active each game. There are 193 students signed up in our class. So we could form a team of 15 in

\[
\binom{193}{15} = \frac{193!}{15!178!} = 8,403,036,288,749,375,695,776
\]
different ways. From this set of 15 we can choose

\[
\binom{15}{13} = \frac{15 \times 14}{2} = 105
\]

teams of 13 to participate in a particular game, and among those 13 we can choose

\[
\binom{13}{5} = \frac{13!}{5!8!} = 1287
\]
teams of 5 actually playing at any one time. And we would still loose against the Utah Jazz!
9.8 Taylor and MacLaurin Series

- named after:
  - Brook Taylor, 1685-1731
  - Colin MacLaurin, 1698-1746

- We want to express a function \( f \) as a power series.

- Idea: Match all derivatives of \( f \) at \( x = 0 \) (MacLaurin Series), or, more generally, at \( x = a \) (Taylor Series).

- The Taylor Series is more general, the MacLaurin Series is a special case of a Taylor Series.

- We get the **Taylor Series**

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

\[
= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \ldots
\]

and, in the special case that \( a = 0 \), the **MacLaurin Series**

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

\[
= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots
\]
• Example (review): $f(x) = e^x$

\[
\frac{d}{dx} e^x \to e^x \to e^x \to - \quad - \quad \frac{f^{(n)}(x)}{n!} = \frac{1}{n!}
\]

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

converges for all $x$

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

• Example $f(x) = \sin x$

$\sin x \to \cos x \to - \sin x \to - \cos x \to \sin x$

$x=0:\quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 1$

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

\[
= 0 + x + \frac{0}{2!} x - \frac{x^3}{3!} + \frac{0}{4!} x^4 + \ldots
\]

• Example $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
• Why does it work?

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n = s(x) \]

\[ f^{(n)}(0) = s^{(n)}(0) \quad \alpha_n = ? \]

\[ \left. \frac{d^m}{dx^m} s(x) \right|_{x=0} = \sum_{n=0}^{\infty} a_n \left. \frac{d^m}{dx^m} x^n \right|_{x=0} = D^m x^n \]

\( n=2 \)

\[ m=1 \quad \left. \frac{d}{dx} x^2 \right|_{x=0} = 2 \times 0 = 0 \]

\[ D = \frac{d}{dx} \]

\( m=3 \)

\[ \left. \frac{d^3}{dx^3} x^2 \right|_{x=0} = 0 \]

\( m=2 \)

\[ \left. \frac{d}{dx^2} x^2 \right|_{x=0} \rightarrow 2 \times \rightarrow 2 \]

\[ \left. x^3 \right|_{x=0} \rightarrow 3 \times 2 \rightarrow 6 \times \rightarrow 6 \]

\[ \left. x^n \right|_{x=0} \rightarrow n \times x^{n-1} \rightarrow n(n-1)x^{n-2} \rightarrow \ldots \rightarrow n! \]

\[ \left. \frac{d^m}{dx^m} x^n \right|_{x=0} = \begin{cases} n! & m = n \\ 0 & m \neq n \end{cases} \]
\[ S(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

\[ s^{(n)}(0) = n! a_n = \frac{f^{(n)}(0)}{n!} \]

\[ a_n = \frac{f^{(n)}(0)}{n!} \]
(Absolute) Convergence of sin and cos series

\[ \sum a_n \leq \sum b_n \]

\[ 0 \leq a_n \leq b_n \]

\[ \sum b_n \text{ converges} \]

\[ \Rightarrow \sum a_n \text{ converges} \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]

\[ \text{absolute conv.} \]

\[ |x| + \left|\frac{x^3}{3!}\right| + \left|\frac{x^5}{5!}\right| + \left|\frac{x^7}{7!}\right| + \cdots \]

\[ \cos x \text{ also converges for all } x \]
Terminology

- The MacLaurin Series for (or “of”) \( f \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

- The Taylor Series for (or “of”) \( f \) about \( a \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

- The “MacLaurin Series” is the same as the “Taylor Series about 0”.

- Computing a Taylor Series for \( f \) is also called “expanding \( f \) into a Taylor Series (about “\( a \”)”.

- A partial sum of a Taylor (or MacLaurin) Series is also called a Taylor (or MacLaurin Polynomial).
• More examples.
• Compute the MacLaurin series for

\[ f(x) = \sinh x \]

in two different ways.

\[
\sinh x \rightarrow \cosh x \rightarrow \sinh x \rightarrow \cosh x
\]

\[
\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 1
\end{array}
\]

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \]

\[
\sinh x = \frac{1}{2} \left( e^x + e^{-x} \right)
\]

\[
= \frac{1}{2} \left[ \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \right) + \left( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \right) \right]
\]

\[
= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots
\]
• Compute the MacLaurin series for

\[ p(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \]

\[ f(x) = 1 + x + x^2 + x^3 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \]

Any polynomial in standard form is actually a MacLaurin Series

\[ f(0) = 1 \]
\[ f'(x) = 1 + 2x + 3x^2 \quad f'(0) = 1 \]
\[ f''(x) = 2 + 6x \quad f''(0) = 2 \]
\[ f'''(x) = 6 \quad f'''(0) = 6 = 3! \]
\[ n > 3 \quad f^{(n)}(0) = 0 \]

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \]

= 1 + x + x^2 + x^3