Reminders

- Do use DSs: today, 3:00pm, Scott
- I am working on the exam. There are a lot of them! Will publish scores and answers probably on Thursday.
- Tomorrow is election day. We will have class, however.
- Of course you know we switched to Mountain Standard Time

Quick Review

- \( \sum_{k=0}^{\infty} a_n \) converges absolutely if \( \sum_{k=0}^{\infty} |a_n| \) converges.
- Absolute convergence implies convergence!
- \( \sum_{n=1}^{\infty} a_n \):
  - Suppose
  \[
  \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho
  \]
Then:

\[
\begin{align*}
\text{if } \rho < 1 & \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges (absolutely)} \\
\text{if } \rho > 1 & \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges} \\
\text{if } \rho = 1 & \text{ then the test is inconclusive}
\end{align*}
\]

• Basically: The series behaves like a geometric series with ratio \( \rho \).
9.6 Power Series

- You can think of a power series as a polynomial with infinitely many terms:

\[ s(x) = \sum_{k=0}^{\infty} a_k x^k. \] (1)

- If the upper limit was \( n \) instead of infinity, we’d have an ordinary polynomial of degree \( n \). In other words, the partial sums of a power series are ordinary polynomials.

- We can now ask the following Questions:
  1. For what values of \( x \) does the power series converge?
  2. Can we get an expression for the sum of the series in terms of \( x \) if the series does converge?
  3. Can we combine (add, subtract, multiply, divide, compose) power series, or differentiate or integrate them, just like polynomials?

- **Definition:** The convergence set (CS for short) of (1) is the set of all real numbers \( x \) for which the power series converges.
Example:

\[
\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if } |x| < 1
\]

- This is an ordinary geometric series.
- The absolute ratio test turns into

\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{|x^{n+1}|}{|x^n|} = |x|
\]

and the series will converge (to \( \frac{1}{1-x} \)) if and only if \(|x| < 1\).
- Of course, we knew that already!
Example:

\[ s(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \]

\[ \text{CS} = (-\infty, \infty) \]

\[ \frac{x^{k+1}}{(k+1)!} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{k!} = \frac{x}{k+1} \]

\[ \lim_{k \to \infty} \left| \frac{x}{k+1} \right| \Rightarrow s < 1 \quad \text{as } k \to \infty \]

\[ \frac{k!}{(k+1)!} = \frac{1 \cdot 2 \cdot \ldots \cdot k}{1 \cdot 2 \cdot \ldots \cdot k \cdot (k+1)} = \frac{1}{k+1} \]
Example:

\[ s(x) = \sum_{k=0}^{\infty} k!x^k \]

\[ | \frac{(k+1)! x^{k+1}}{k! x^k} | = | (k+1) x | \rightarrow \infty \]

unless \( x = 0 \)

\[ \left| (k+1) x \right| = \left| k+1 \right| x \rightarrow a \]

\( k \rightarrow \infty \)

constant

\[ x^{k+1} = x \cdot x^{k+1-k} = x = x \]
These examples sum up all possibilities:

- The convergence set of

\[ s(x) = \sum_{k=0}^{\infty} a_k x^k \]

is

(i) the single point \( x = 0 \), or
(ii) an interval of the form \((-R, R)\), with \( R > 0 \), plus possibly one or both end points, or
(iii) the set of all real numbers.

- In the cases (i), (ii), (iii) we say that the radius of convergence of the power series is 0, \( R \), or infinity, respectively.

Every power series has a convergence set of one of those three types. There are no other possibilities.

This is not obvious, and, I think, actually counter-intuitive!

- It is worth our while to study this fact closely, to see why, and not just, that it is true.
- So here is the proof from the textbook.
- Suppose

\[ s(x) = \sum_{n=0}^{\infty} a_n x^n \]
converges for

\[ x = x_1 \neq 0. \]

• Then

\[
\lim_{n \to \infty} a_n x_1^n = 0
\]

and there exists a number \( N \) such that

\[ |a_n x_1^n| < 1 \]

for all \( n > N \).

• Suppose that

\[ |x| < |x_1|. \]

Then

\[
|a_n x^n| = |a_n x_1^n| \left| \frac{x}{x_1} \right|^n < \left| \frac{x}{x_1} \right|^n < 1.
\]

• Now note that

\[
\sum \left| \frac{x}{x_1} \right|^n
\]

is a geometric series with ratio

\[ \left| \frac{x}{x_1} \right| < 1. \]
It converges, and by the ordinary comparison test so does
\[ \sum_{n=0}^{\infty} |a_n x^n|. \]

- So, if \( s(x) \) converges for \( x = x_1 \), it also converges for every \( x \) with
  \[ |x| < |x_1|. \]

- By the same token, if it diverges for \( x = x_2 \) then it diverges for all \( x \) with
  \[ |x| > |x_2| \]
  since if it did converge for \( x \) it would also have to converge for \( x_2 \).

- This establishes our statement about the three kinds of convergence sets.

- Note that we actually established absolute convergence.

- **A power series converges absolutely in the interior of its convergence set!**

- Let’s do more examples!
\[ s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}. \]

\[
\left| \frac{x^{n+1}}{n+1} \right| = \left| \frac{x^n}{n} \right| 1 \times 1 \rightarrow s < 1 \\
\Rightarrow 1 \times 1 < 1 \\

\frac{1}{2}, 2, 3, 4, 5, 6, 7, 8, \ldots \Rightarrow 1
\]

\[
\operatorname{LH} \quad \frac{n}{n+1} \rightarrow \frac{1}{1} = 1
\]

\[
\begin{aligned}
\text{when } x = 1 & \quad s(x) = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \\
\text{when } x = -1 & \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \text{converges}
\end{aligned}
\]

\[ C = \left[ -1, 1 \right) \]
\[ s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \times 2^n}. \]

\[
\left| \frac{x^{n+1}}{(n+1) \times 2^{n+1}} \right| = \left| \frac{n \times x}{(n+1) \times 2} \right| = \left| \frac{n}{(n+1) \times 2} \right| \times 1
\Rightarrow \frac{1}{2}
\]

\(-2 < x < 2\)

\[
\left| \frac{1}{2} \times x \right| < 1
\]

\[
|x| < 2
\]
Shift of Origin

- The point \( x = 0 \) is special for a power series. It is the center of the convergence set.

- Of course, we can do a \textbf{shift of origin}, or a \textbf{change of variable}:

- The series

\[
s(x) = \sum_{n=1}^{\infty} a_n (x - a)^n
\]

is a power series in \( x - a \), centered at \( x = a \).

- Example:

\[
s(x) = \sum_{n=2}^{\infty} \frac{(x + 2)^n \ln n}{n \times 3^n}.
\]

- Find the convergence set.

\[
\left| \frac{(x+2)^{n+1} \ln(n+1)}{(n+1) \times 3^{n+1}} \right| = \frac{\ln(n+1)}{3 \ln n} \leq \frac{(x+2)^n \ln n}{n \times 3^n}.
\]

\[
\frac{|x+2|}{3} \leq 1 \Rightarrow -3 \leq x + 2 \leq 3 \Rightarrow -5 \leq x \leq 1.
\]
\[
\frac{\infty}{\infty} \frac{\ln(n+1)}{\ln(n)} \Rightarrow \frac{1}{n+1} = \frac{n}{n} \quad \text{as} \quad n \to \infty
\]
• Example:

\[
s(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!}
\]

\[
= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]